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Theory and practice of origami in science

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Theory and practice of origami in science†

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No longer just the purview of artists and enthusiasts, origami engineering has emerged as a potentially powerful tool to create three dimensional structures on disparate scales. Whether origami (and the closely related kirigami) engineering can emerge as a useful technology will depend crucially on both fundamental theoretical advances as well as the development of further fabrication tools.

Origami and kirigami are ancient art forms that aim to turn a flat sheet of paper into a three dimensional sculpture (Fig. 1). Because of new fabrication methods to design responsive materials, recent years have seen a surge of interest in using origami-related ideas in engineering to create three dimensional structures from thin, and initially flat, films[1–4]. The needs of engineering have similarly led to renewed interest in understanding the fundamental physics and mathematics that underlies folding.

Two factors have led to this interest. First, mathematical tools have led to new and powerful ways to design complex sculptures[5–7]. The fold pattern of these sculptures are, at least in part, designed by a computer, and it has become clear that the world of origami is far richer than one might have been led to believe. In principle, any shape can be approximated from a single flat sheet[7]. Second, new fabrication techniques have enabled the creation of "architected materials," composite materials whose local structure is designed to lead to new effective properties at longer ranges[8–10]. The incorporation of responsive materials into these architectural structures, from swelling hydrogels to liquid crystal and dielectric elastomers, allows us to shape differential strains within structures and thereby get them to fold into three dimensions[3,11,12].

Success could revolutionize the manufacture of devices from macroscopic scales down to millimeter or even micron scales. Because origami structures are patterned while flat, self-folding devices are amenable to lithographic or roll-to-roll processing. This suggests that, with the right processing and materials, self-folding origami structures can be produced in large batches. Since the mechanical principles driving self-folding are universal and scale-free[13,14], the techniques developed can be applied, at least in principle, on a variety of materials and processes. One of the holy grails of soft matter research has been to develop techniques to self-assemble complex three-dimensional structures. Self-folding origami is a potential alternative to these traditional self-assembly pathways; by placing functional elements on the initially flat structures and folding them into three dimensions, one could achieve the same functionality.

Quite a bit of progress in origami engineering has so far followed an "art-mimetic" approach: one borrows a design from the world of origami art and adapts it to a new material and new applications[15,16]. This is, of course, a fruitful way to proceed, especially considering the wealth of experience from the world of origami art. Despite the wealth of examples of this type, this perspective will take a somewhat idiosyncratic view of origami science. I will argue that fulfilling the full promise of origami engineering will require theoretical and experimental progress, proceeding in tandem. Following this way of thinking, I will orient the discussion from the side of origami mathematics[17].

1 How should scientists define origami?

The traditional conception of what defines origami is not necessarily applicable to science. A traditional origami artist might impose arbitrary rules on themselves – for example, fold a single sheet of paper without cutting – which is far too rigid to be useful from the perspective of science. Similarly, two techniques that an artist might describe as "origami" may lead to very different underlying mathematics. Here we will take a different approach. In order to avoid the inevitable splitting of hairs that comes with the making of strict definitions, I will provide four assumptions, none of which can be met experimentally, but which can serve the basis to unite many types of origami and kirigami research into a broader field of mathematics that can inform how we think about origami structures. To that end, here are four characteristics of *ideal* origami:

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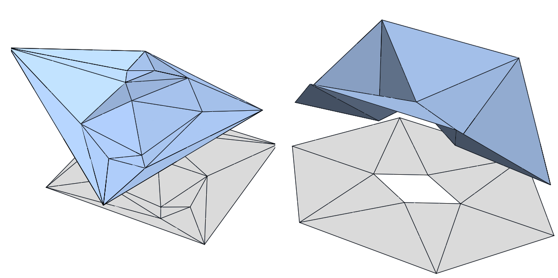
1. **An origami structure is constructed from vertices, edges and faces which are zero-, one-, or two-dimensional respectively**. This means, in principle, that vertices are point-like and edges and faces are infinitely thin. This assumption is certainly not satisfied in practice and some effort has gone in to understanding what happens when these assumptions are lifted. For example, there has been theoretical work trying to understand the folding of "thick" origami [18–21].

2. **Origami edges do not curve and faces do not bend.** Again, this is probably violated in practice, and excludes work on curved folds [22–24], and face bending is particularly easy to see in an origami structure [8]. Nevertheless, if the folds remain straight then any interior face can only bend along a face diagonal anyway; this suggests that face bending origami can, in principle, be modeled by introducing auxiliary folds across the diagonals. Thus, there is little loss of generality with such an assumption.

3. **Edges do not stretch.** While not true experimentally, in the limit that the ratio of bending to stretching stiffnesses goes to zero, this assumption is not difficult to satisfy approximately. In practice, this can be achieved when the thickness is small and any external forces are sufficiently weak.

4. **There is no gluing, cutting, or rearranging of folds and vertices.** While we allow origami to explicitly allow the possibility of holes and cutting, we will exclude "gluing." This gluing assumption is often explicitly lifted by "kirigami" [25], though this usage kirigami does not at all conform to how artists use those words and does not always fit the scientific literature either. On the other hand, this assumption is fairly easy to achieve experimentally.

No experiments can meet assumptions 1 - 4, though many do come close and can, at least, be accurately modeled by making these assumptions. For example, real paper is not infinitely thin, and experience trying to fold a particularly tricky piece of origami demonstrates that paper can crumple and deform in confounding ways. Yet, paper is very thin compared to the lateral dimensions of most origami, and fold lengths and faces do tend to be rigid under careful folding. In that sense, it is clear that there are conditions in which real origami is at least close to ideal origami.



(*a*) (*b*)

The above assumptions also contain a few surprising cases that one might not traditionally think of as origami. As one example, triangular meshes, which are used to model elastic membranes, when elastically thin so that they bend far more easily than stretch, can be an example of an origami mechanism. As a second example, note that there is no assumption that vertices should be flat. In some versions of kirigami, one generates vertices with nonzero Gaussian curvature by removing wedges of material and reattaching the paper along the newly cut seams [25]. The process of cutting and gluing is excluded from assumption 4, but the structures formed after gluing and cutting are not. It is, perhaps, a subtle point: assumption 4 is there to exclude origami that changes its structure dynamically as it deforms.

Among flat, ideal origami structures, two cases have been very well-studied. The first is quad origami, in which all faces are quadrilaterals and each vertex has four folds associated to it. This case is a generalization of the celebrated Miura ori fold pattern [26]. Four-fold vertices have precisely one degree of freedom and the relationships between the dihedral angles can be determined analytically. The second important case is triangulated origami, in which all faces are triangles but vertices can have an arbitrary number of folds emerging from them. In this case, faces are always planar and the length constraints associated with folds ensure that the planar angles between adjacent folds at each vertex stay the same. This allows one to understand the kinematics of triangulated origami in terms of the kinematics of linkages.

The assumptions 1 - 4 bring the study of origami squarely within a venerable historical context going back at least as far as Cauchy [27] and J.C. Maxwell [28,29], who have studied the rigidity of frameworks and linkages. Indeed, origami and kirigami mechanisms conforming to the assumptions above can sometimes be mapped directly to the mechanics of a specific class of frameworks. This allows us to bring some powerful mathematical tools to bear on the mechanics of origami. Similarly, others have explored the folding of crystalline lattices [30–33] and highlighted some fundamental connections between random folding and spin systems [34,35] and coloring problems [36]. These two bring additional insights to the kinematics of origami structures and can be used to shed light on the behavior of real origami systems that approximate assumptions 1–4.

1.1 Infinitesimal isometries

We start our analysis of the motions of origami by assuming all faces are triangles. This is, in some sense, the "base" case: any origami fold pattern can be obtained by selectively rigidifying some of the folds of a triangulated fold pattern. As will become clear, we must distinguish the case in which the origami fold pattern is flat from the case in which the initial origami structure

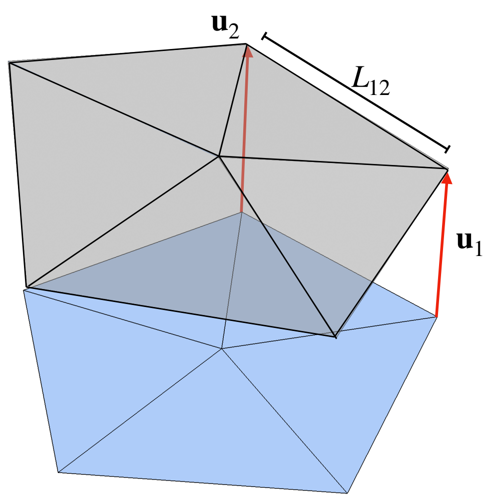


Fig. 2 Origami can be represented as a mechanical linkage with vertices, indexed by integer n, displaced by $\mathbf{u}_n$. The blue origami is flat while the gray is folded.

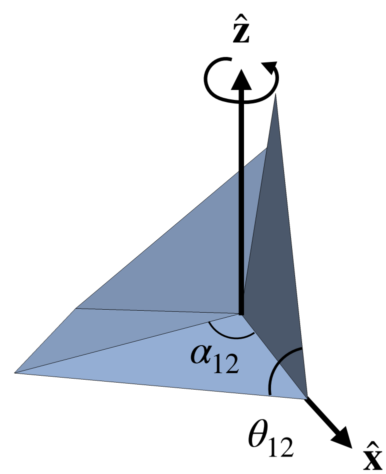


Fig. 3 Rotations of a vertex can be constructed by the composition of alternating rotations about the $\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}$. To do so, start with one fold along the $\hat{\mathbf{x}}$ axis, successive rotations of the origami about the $\hat{\mathbf{z}}$ axis and $\hat{\mathbf{z}}$ must, eventually, bring us back to the original configuration.

of equations – one for each edge – of the form

$$(\mathbf{X}_n - \mathbf{X}_m)^2 = L_{nm}^2, \tag{1}$$

where $\mathbf{X}_n$ is the three dimensional position of vertex n and L_{nm} is the equilibrium length of the edge joining vertex n and m (Fig. 2). If one writes a vector, $\mathbf{u}$ concatenating the displacements of all of the vertices in three dimensions, Eq. (1) determines the displacements that preserve edge lengths to first order,

$$\mathbf{R}\mathbf{u} = \mathbf{0}. \tag{2}$$

The matrix $\mathbf{R}$ has a row for each edge and *three* columns for each vertex.

How many degrees of freedom does a generic, triangulated origami structure support? To make an appropriate count, we must distinguish between vertices and edges in the interior, which are actual vertices and folds of the origami fold pattern, and those on the boundary. Let V_i and V_b denote the number of interior and boundary vertices respectively, and let E_i and E_b denote the number of interior and boundary edges. Then a simple counting argument shows that the solution space of Eq. (2) is generically of dimension $V_b + 3$. To obtain this count, we need two relations. The first is Euler's formula for a planar graph, $F - E_b - E_i + V_b + V_i = 1$. The second is the relation for triangulated origami that each face abuts three edges but that, consequently, internal edges are double counted. Therefore, $3F = 2E_i + E_b$. Thus, the number of parameters needed to place $V_b + V_i$ vertices in three dimensions is $3V_b + 3V_i = 3 + 3E_b + 3E_i - (2E_i + E_b) = 3 + 2E_b + E_i$. Finally, the dimension of the configuration space is given by $D = 3V_b + 3V_i - E_i - E_b$ since each edge provides one constraint, leaving $D = 3 + E_b = 3 + V_b$ since $E_b = V_b$ for a polygon. However,

interior vertex can be displaced vertically without changing the edge lengths *to first order*. It turns out, however, that there are V_i quadratic constraints in this case[37], strongly suggesting $D = V_b + 3$ is still the correct dimension of the configuration space. This is an upper bound for the dimension of the origami configuration space, in general. If we proceed to rigidify a single fold in order to make a face with more than three sides, we must then add a linear constrain to the infinitesimal motions, reducing the number of degrees of freedom by one.

1.2 Nonlinear approaches to modeling

An alternative approach to modeling is to consider constraints on a single origami vertex[38,39]. Consider, for example, the intersection of a single vertex with an imaginary sphere whose center is placed on the vertex. This intersection traces out a spherical polygon on the sphere's surface. Now align one of the folds with the $\hat{\mathbf{x}}$ axis and the face joining fold 1 to fold 2 with the $xy-$plane. To get from the first to the second fold, one must then rotate the vertex about the $\hat{\mathbf{z}}$ axis by the first interior angle, α_{12}, so that the second fold is now aligned with the $\hat{\mathbf{x}}$ axis. A second rotation can be performed about the $\hat{\mathbf{x}}$ axis by the fold angle θ_2. The composition of these rotations must then bring us back to our original configuration. Specifically,

$$\mathbf{R}(\theta_1, \hat{\mathbf{x}})\mathbf{R}(\alpha_{N1}, \hat{\mathbf{z}}) \cdots \mathbf{R}(\theta_3, \hat{\mathbf{x}})\mathbf{R}(\alpha_{23}, \hat{\mathbf{z}})\mathbf{R}(\theta_2, \hat{\mathbf{x}})\mathbf{R}(\alpha_{12}, \hat{\mathbf{z}}) = \mathbf{1}, \tag{3}$$

where $\mathbf{R}(\theta, \hat{\mathbf{n}})$ denotes the rotation matrix rotating by an angle θ about axis $\hat{\mathbf{n}}$. This method is closely allied with the analysis of origami single vertex rigidity using spherical trigonometry[40].

A system of constraints of this type allow one to generate nectearing or bending) at all is NP-hard[41]. And suppose an origami fold pattern can be folded. Then the number of ways of folding the origami structure is generically exponential in the number of vertices[37,42,43].

For triangulated origami, for example, the infinitesimal isometries all have the form of vertical displacements of the $V_b + V_i$ vertices. Denoting the vertical displacement of the n^{th} vertex by h_n, Bryan Chen and I have shown that the heights satisfy a system of quadratic equations,[37]

$$\sum_{n,m} h_n h_m Q_{nm,I} = 0, \tag{4}$$

for each internal vertex I. There are, therefore, V_i equations. For origami with one degree of freedom (having $V_b = 4$), we use numerical methods to find that, surprisingly, there are always 2^{V_i-1} solutions. This is a purely numerical result and implies that all solutions of Eq. (4) are generically real. Whether it holds more generally is unclear; what is clear is that a single fold pattern, at least for triangulated origami, often has many ways of being folded.

2 Origami Design

Some of the most important scientific and mathematical work on understanding what is possible with origami comes from artists and mathematicians. There are several important results that highlight both the potential and challenges of origami design. It is known that a single sheet of paper can be folded into *any* polyhedron, as will be described in the next section. This is the basis behind the statement that any shape can be approximated. However, the number of folds required can be quite large; there is no sense in which the computer program produces a design that is optimal. Design within a more restricted set of patterns is, however, possible.

2.1 Designing shapes

The design of fold patterns that produce specific, desired shapes is one of the oldest in origami[44]. Computation design tools have been developed by several authors, most notably Robert Lang, who developed a program called "Tree Maker"[5,45]. Tree Maker allows a user to design an origami base – essentially by specifying a tree graph that can serve as part of the body plan of a more complex origami design. The program then proceeds to provide a fold pattern which the artist can use to fold the basic body plan, after which the artist adds additional folds to further shape the structure.

One might wonder whether one can actually fold a single sheet of paper into any shape. In fact, this question has been answered in the affirmative. First, Demaine has proven that it is possible to approximate any voxelated shape by folding alone using a single, universal fold pattern[46]. The technique takes advantage of an origami design motif called box pleating[47]. Second, further

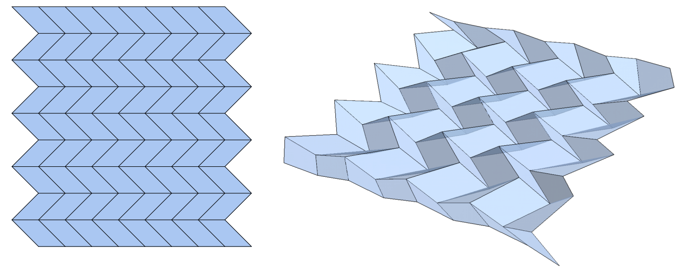


Fig. 4 The Miura ori fold pattern and one of its three dimensional configurations.

Neither of these design tools are practical, however. In particular, because they are approximating polyhedra with vertices that have Gaussian curvature using a sheet whose vertices do not, some of the area of the original sheet must be tucked away and hidden, essentially by forcing the origami faces to nearly overlap, and there is no sense in which the computations are optimal from the point of view of minimizing hidden area. Secondly, techniques such as box pleating for producing voxelated designs are computationally hard[47].

One possible approach to simplifying the origami design problem is to limit the search space to a smaller family of possible fold patterns. These methods are typically variations on the Miura ori design[49] (see Fig. 4). Designs based on quadrilateral faces, such as the Miura ori, are usually called "quad meshes." The difficulty with quad meshes as a design motif is that they are not necessarily rigidly foldable. The counting arguments of the previous section show this: each vertex has four folds and, consequently, one nontrivial degree of freedom. Yet, each closed loop of folds must be compatible in order for a folding motion to be possible. The high symmetry of the regular Miura ori pattern in Fig. 4 introduces enough redundancy that all of these additional constraints can be satisfied. For arbitrary meshes, however, they could not generically be. One solution is to introduce folds along a diagonal of each face. For an infinite structure, however, there are still only enough degrees of freedom to balance the constraints; the generic case is still rigid.

To rectify this, Tachi has developed equations that can determine whether quad mesh origami is, indeed, rigidly foldable[51]. These developments, coupled with geometric considerations, have allowed several groups to design new quad mesh origami. In this vein, Dudte *et al.* and others have shown that quad mesh origami can be designed to fold into a great variety of curvatures by designing a precise spatial variation in the quad mesh origami[50,52,53]. Some designs are shown in Fig. 5.

2.2 Designing mechanical response

In contrast to shape, even less is known about designing the me-

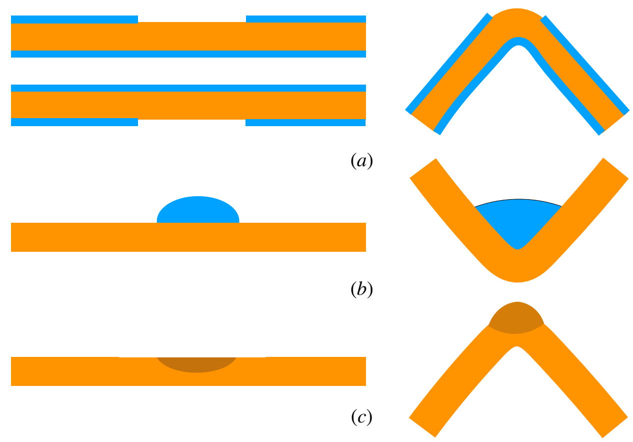


Fig. 6 Schematic methods to fabricate self-folding structures. (a) Tri-layer or bilayer systems with swellable gel (orange) sandwiched between stiffer layers (blue). (b) Contracting or swelling liquid at folds. (c) Material inhomogeneity driving differential swelling at a fold.

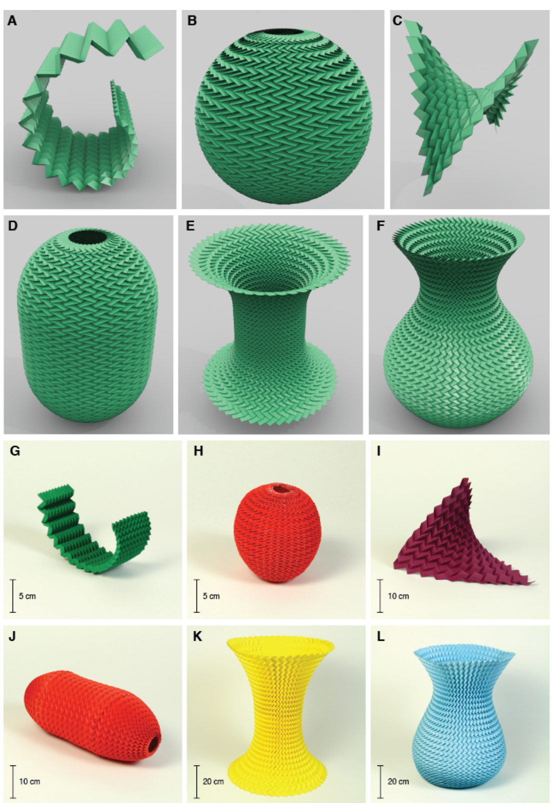


Fig. 5 Reproduced from ref.[50] with permission from Springer Nature, copyright 2016. Note that, in this work, quadrilateral faces are allowed to stretch slightly and bend but that this can often be accommodated by triangulated the design first.

to different, and tunable, fold configurations[56]. For example, the Miura ori with different configurations of "pop-through" defects have a rationally tunable mechanical response[8,57].

Beyond this, there is a great deal of experience, from traditional origami artists, on how different fold patterns respond to deformations. Designs exist for bistable and self-locking origami[58], and some design approaches exist within certain classes of origami structures[59]. While there have been some successes using topology optimization to design mechanical response[60], but we have only begun to scratch the surface of what is possible.

3 Self-folding

The driving impetus for the emergence of origami as a tool of engineering comes from the fabrication of structures that can fold themselves up from an initially flat, thin film. There seems to be two main mechanisms for driving the self-folding of origami. One can draw the folds to bend by adding small droplets of an evaporating or contracting liquid, or one can fabricate materials that develop differential strains cross their thickness. Because mechanisms of differential strain are, essentially, scale-free, materials have been developed on disparate length scales from the human-scale to a scale of tens of microns.

The mechanism for self-folding origami devices is primarily one of differential growth. This is often achieved by bonding together materials with different proclivities toward expansion under changes in temperature, pH, or some other change of environmental conditions. This has taken the form of graphene-glass bimorphs[61], polymer trilayers with a swellable hydrogel sandwiched between two stiffer layers[11], shape memory composites[62], or contracting solder[2]. In most of these cases, the folding generates a preferred fold angle on several of the folds.

In the typical case of modeling the self-folding process, one

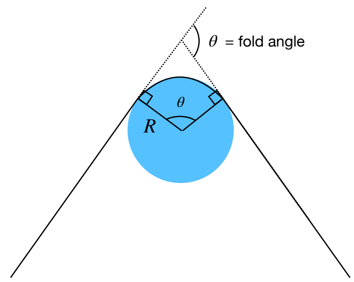


Fig. 7 Folding a Hookean elastic sheet at fixed width leads to an energy in the form of a torsional spring.

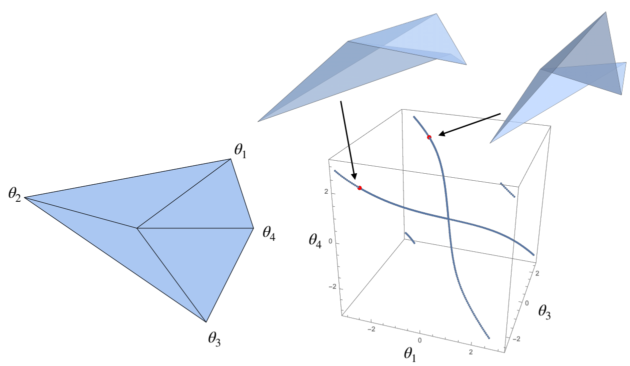


Fig. 8 Three fold angles of a degree four vertex. Note that the calculated angles are periodic in the interval $(-2\pi, 2\pi]$ and the origami structure can, in principle, self intersect.

where f indexes the folds of an origami fold pattern. There is a connection between this form of a fold energy for discrete structures and in the continuum. The bending of an elastic sheet is usually modeled by an energy of the form,

$$E_{cont} = \frac{1}{2}\kappa \int dA\,(H - H_0)^2, \tag{6}$$

where H is the mean curvature, H_0 is a spontaneous curvature, and κ is a rigidity that scales with the cube of the sheet thickness. If we imagine that a fold is a portion of an elastic sheet that has been rolled into a cylinder of radius R to over an angle θ. The width of the fold is, therefore, $w = R\theta$ so that $R = w/\Delta\theta$ (Fig. 7). Substituting this back into the energy,

$$E_{cont} = \frac{\kappa}{8w} L\,(\theta - H_0 w)^2. \tag{7}$$

This is precisely of the form in Eq. (5) when the folding width is fixed. The preferred fold angle is, therefore, given by $\bar{\theta} = H_0 w$ and the fold stiffness k_f is proportional to the fold length, L. In real materials, one expects deviations from this simple form[63,64].

The best understood cases of self folding are single vertices and,

tion of a fourth (Fig. 8). If one allows for the faces to pass through each other, the two branches are actually connected to each other and only become distinct when self-intersections are disallowed (though proving this can be a challenge[67,68]). One can superimpose, on Fig. 8, the energy of Eq. (5), whose equi-energy surfaces are ellipsoids. The origami will fold along one branch or the other so long as the energy decreases along that branch. One can now formulate the question: when is the energy decreasing only along one origami branch? The answer, according to Tachi and Hull, is that the gradient of the energy in the flat state should be perpendicular to all the "wrong" branches[69] rather than being directed along the correct branch. In fact, this arises from the fact that the branch structure near the flat state is always invariant under flipping the sign of all the fold angles (after all, flip the origami over). If the gradient isn't perpendicular to a branch, the energy necessarily must decrease along one direction of the branch. The gradient of the fold energy is

$$\left.\frac{\partial E}{\partial \theta_f}\right|_{\theta=0} = -k_f \bar{\theta}_f, \tag{8}$$

meaning one can use both the stiffness k_f and prescribed angle $\bar{\theta}_f$ to tune the gradient.

This is, unfortunately, asking a lot. For triangulated origami with no holes, the number of folds is

$$E_i = V_b + 3V_i. \tag{9}$$

As the origami complexity increases, the number of branches grows exponentially whereas the number of folds grows only linearly. There are not enough parameters to ensure that the perpendicular condition of Tachi and Hull can always be satisfied! The problems are only exacerbated with non-triangulated origami, which must have strictly fewer folds than in the triangulated case.

Yet this may not be the end of the story. In many cases, the branches are governed by individual vertex buckling. Recently, Hayward *et al.*[70] have found that origami folding can be controlled by engineering vertices that are biased to buckle upward or downward independently of the folds. Stern *et al.*[71] have also found methods that can control the bifurcations of mechanical linkages and, in particular, origami. These new results show clearly a need for further analysis in order to understand and fine-tune the folding of origami.

4 Where are the emerging challenges?

As we have progressed, we also have developed a better picture of the most difficult challenges that must be solved before this becomes a technology. There are, first of all, the challenges we know. Given the proliferation of folding pathways for even simple origami designs in tandem with the difficulty of determining if a pattern is even foldable, new methods to ensure robust folding will certainly need to be developed, and the interplay between energetics and kinematics will need to be better understood. And mine the effective mechanical properties of folded figures. The same isometric motions that allow origami to fold also ensure that there is a pathway toward easy unfolding. To deal with this, packagers either use adhesives (which experience tells us can be a struggle to open) or mechanisms that lock, for example through flaps. The degree of geometrical control required to do this is still beyond what any experiments have achieved. To do this would likely require dynamically altering how a system folds, perhaps by activating folds sequentially[72]. Only then could one create a structure that folds on its own and, subsequently, becomes rigid.

There are also unknown challenges. Most examples of self-folding origami structures are still at the "proof-of-principle" stage. The pathway to get from this stage to one where folding is fast and the resulting mechanism performs robustly remains unclear, though some work has been attempted along these directions. How easily can self-folding materials be coupled with the kinds of materials one builds electronic or optical devices from[73]? Can origami mechanical structures carry a load? To what extent can the folding pathways be programmable or, even, reprogrammable[74]? What kind of yield could one expect and how complex could the origami be?

Finally, and importantly, more realistic models of origami folding will need to be developed to better understand the behavior and buckling of origami devices in the real world. Here I am envisioning something between the idealized mathematical approaches that have been mostly studied until now and more accurate but computationally expensive finite-element simulations[75,76]. How would a typical origami structure deform? What parts of the structure determine the rigidity, for example?

The field of origami is still in its infancy, yet it could revolutionize the manufacture and fabrication of three dimensional structures. We are at the point now where, perhaps, we could start asking some of these more difficult questions.

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Conflicts of interest

There are no conflicts to declare.

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