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# Adhesive contact of a rigid circular cylinder to a soft elastic substrate – the role of surface tension

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## Abstract

This article studies how surface tension affects the adhesive contact mechanics of a long rigid cylinder on an infinite half space comprising of an incompressible elastic material. We present an exact solution based on small strain theory. The relation between indentation force and contact width is found to depend on a single dimensionless parameter $\omega = \frac{\sigma}{4(\mu R)^{2/3}(W_{ad}/2\pi)^{1/3}}$, where $R$ is the cylinder radius, $W_{ad}$ the interfacial work of adhesion, $\sigma$ and $\mu$ are the surface tension and shear modulus of the half space respectively. The solution for small $\omega$ reduces to the classical Johnson-Kendall-Roberts (JKR) theory whereas for large $\omega$, the solution reduces to the small slope version of the Young-Dupre equation. The pull-off phenomenon is carefully examined and it is found that the contact width at pull-off is reduced to zero when surface tension is larger than a critical value.

## 1 Introduction

A number of recent studies have demonstrated that surface tension can act as a significant and even dominant agent in the mechanics of compliant materials such as elastomers and gels. Examples include instabilities[1–3] and deformation driven by surface tension[4,5], violation of Young's equation for wetting[6–12], attenuation of driving forces for crack or void growth[13–16], and breakdown of conventional adhesive contact mechanics[17–21].

The last example cited above, i.e., the adhesive contact between non-conforming surfaces, is an important basic problem in adhesion science. For example, consider the contact of a rigid sphere with the flat surface of an elastic substrate. For small deformation, where the contact radius is small in comparison with the radius of the sphere, the solution is given by Hertz[22]. Hertz theory assumes adhesion-less contact; as a result, the traction on the contact region is compressive everywhere and vanishes at the contact line. In 1971, Johnson, Kendall and Roberts (JKR)[23] extended Hertz theory to allow for adhesion. The JKR theory has been extremely successful in describing the adhesive contact of elastic spheres[24]. However, recent observations of contact deformations on soft substrates, such as plasticized polystyrene[25], hydrogels[26] and silicone gels[17], caused by adhesion of hard microparticles or nanoparticles deviate considerably from JKR theory. For example, Style *et al.*[17] have reported that the exponent of power-law relationship between the contact radius $a$ (indentation depth $\delta$) and sphere radius $R$ changes from $a \propto R^{2/3}$ ($\delta \propto \sqrt{R}$) to $a, \delta \propto R$ as the sphere reduces in size or the substrate becomes softer. The transition in scaling observed in these experiments has been interpreted as a corresponding underlying transition from the JKR limit where deformation is resisted primarily by bulk elasticity, to the "liquid" limit where deformation is resisted primarily by surface tension. This transition in scaling is consistent with the molecular dynamics simulations by Cao *et al*.[27]. Using a large deformation Finite Element Model which incorporates both surface tension and non-linear elasticity, Xu *et al*. [18] have demonstrated the transition from the elasticity dominated regime where $a \propto R^{2/3}$ to the surface tension dominated regime where $a \propto R$ depend on a single elasto-capillary number $\alpha \equiv \sigma / 4\mu R$ - small $\alpha$ favors JKR theory whereas large $\alpha$ favors surface tension. Their numerical results are found to be in good agreement with the experiments reported by Style *et al*[17]. Xu *et al*.'s results are for rigid spheres that are subjected to zero load. The case of finite load has been recently considered by Hui *et al*.[19], who provided analytical expressions for the relations between indentation depth (contact radius) and applied load. They also examined the effect of surface tension on the pull-off load as a special case of their theory. It should be noted that similar transitions are predicted for soft particles on a rigid surface, as demonstrated by the theoretical works of Carrillo and Dobrynin[20] and Salez et al[28]. The dimensionless parameter $\omega = \frac{\sigma}{4(\mu R)^{2/3}(W_{ad}/2\pi)^{1/3}}$ which controls the transition between JKR and wetting limit was first suggested by Carrillo *et al.*[29] when they studied the adhesion of a soft nanoparticles *to a rigid substrate.*

Another geometry that is of practical interest is a circular rod or circular cylinder. This geometry arises whenever cylindrical or "hair-like" deformable objects adhere to each other or to a substrate. As explained in more detail below, the behavior of this two-dimensional problem differs qualitatively from the better studied three-dimensional (axisymmetric) problem of contact between a sphere and a flat substrate. A Molecular Dynamic simulation combined with theory was performed by Cao *et al*.[30] to study the effect of surface energy on contact of spherical and cylindrical nano-particles with a soft solid without indentation force. The adhesive contact mechanics of an infinitely long rigid cylinder with an elastic half space in the absence of surface tension can be found in the works of Barquins[31] and Chaudhury *et al*.[32], while the two-dimensional Hertzian contact problem with surface tension has been studied by Long *et al*.[33].

In this paper a different approach is used to include both adhesion and surface tension. Our derivation below follows the approach of JKR[23], that is, the adhesive force is confined to a region small compared with typical specimen dimensions, so that the contact edge can be viewed as the tip of traction free crack. In reality, adhesive forces acting outside the contact zone will serve to place a limit on the maximum stress that can be reached. To address this issue, Derjaguin, Muller, and Toporov[34] proposed a theory (DMT theory) that takes these forces into account, but assumes that they do not change the shape of the surface determined by the

Hertz theory. This approximation leads to tensile stresses that are finite outside the contact line but are zero inside. A unifying theory of elastic contact of spheres was proposed by Maugis[35], who used a Dugdale-Barenblatt cohesive zone model to represent the surface forces outside the contact zone. This model removes the stress singularity of the JKR theory, and it also removes the stress discontinuity of the DMT theory. For the case of a rigid sphere on an incompressible elastic substrate, Maugis's result[35] showed that the transition from JKR to DMT theory depends on a single dimensionless parameter

$$\lambda = \sigma_c \left( \frac{R_s}{\mu^2 W_{ad}} \right)^{1/3} , \quad (1)$$

where $\sigma_c$ is the cohesive stress, $\mu$ the shear modulus of the substrate, $R_s$ the radius of the sphere and $W_{ad}$ the work of adhesion. Typically, JKR theory is valid for $\lambda > 1$ and this is the typical situation in experiments using elastomeric spheres ($\mu \approx 10^6 Pa$) with radius around 1mm[36]. For very soft substrate where surface tension effect are important, $\mu < 10^5 Pa$. Since $\lambda \propto \left( R_s / \mu^2 \right)^{1/3}$ the reduction of modulus will extend the region of validity of the JKR model to micron size spheres. Indeed, Style *et al*'s experiments[17] showed that JKR theory works well for glass spheres with radii ranging from 3 to 30 microns on a silicon substrate with $\mu = 170 kPa$. In light of this, it is reasonable to adopt the JKR approach to study surface tension effect in the contact mechanics of soft materials. Finally, we note that the unified approach of Maugis[35] was also used by Baney *et al*.[37] and Leng *et al*.[38] to study two dimensional contact problems involving cylindrical geometry.

## 2. Statement of Problem and Summary of Approach

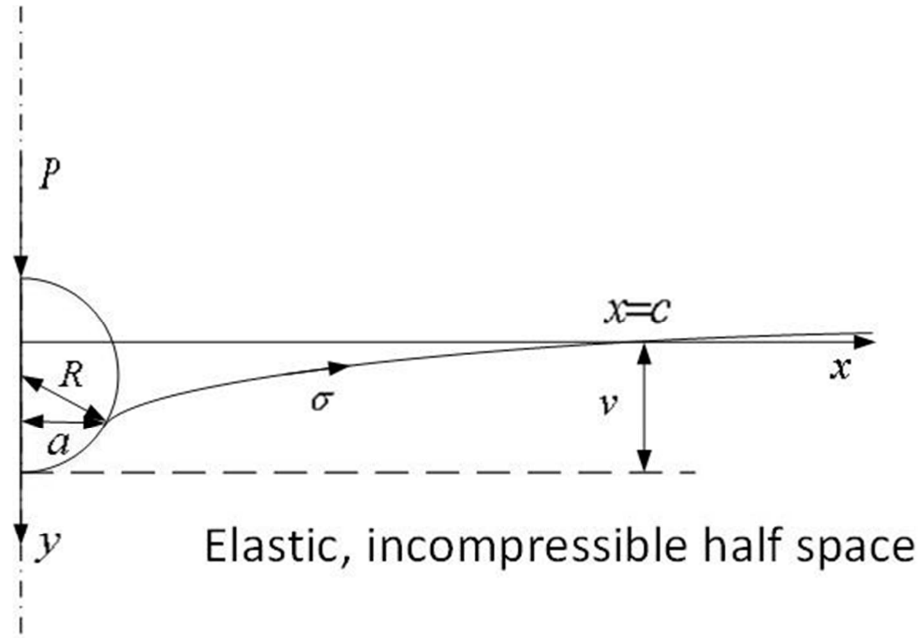


Fig. 1 A long rigid cylinder of radius *R* and length *L* >> *R* indents an isotropic and incompressible elastic half-space under the influence of a vertical line force *P*. Indentation is resisted by the elasticity of the substrate as well as the surface tension $\sigma$ of the elastic substrate.

Fig. 1 shows a rigid cylinder of radius *R*, with length *L* >> *R* in the out-of-plane direction, brought into contact with the flat surface of a substrate by a vertical line force *P* (units = force/unit length, so total force acting on the cylinder is *PL*). Let (*x,y,z*) denote the position of a material point in the elastic half-space. Before deformation, the bottom of the cylinder touches the substrate surface at the origin, x = y = z = 0. The substrate occupies the half space y > 0 and is assumed to be linearly elastic, isotropic and *incompressible*, with shear modulus $\mu$ and Poisson's ratio *v* = 0.5. The contact zone is a long rectangular strip of width 2a and length *L*. We also assume that the surface stress tensor is isotropic, so it can be represented by the scalar surface tension $\sigma$. Just as in JKR theory, our analysis is based on small strain theory.

In contrast to the JKR theory, which assumes frictionless contact, we assume a no-slip contact condition. That is, a material point on the substrate surface is held fixed once it comes into contact with the rigid cylinder. Without the no-slip condition, one would have to introduce a new parameter into the model – the interfacial tension in the contact region. In general, the no-slip contact condition will cause non-zero shear traction in the contact region. However, a classical result in linear elasticity states that the no-slip boundary condition is consistent with a vanishing shear traction provided that the cylinder is rigid and the substrate is incompressible and infinite in extent[22]. Fortunately, the bulk modulus of most elastomers and hydrogels is much higher than their shear moduli, so incompressibility is an excellent approximation.

As in JKR theory, our basic approach is based on energy balance, that is, the relation between the contact radius and the applied load is obtained by setting the energy release rate $G$ of the external crack (the air gap between the cylinder and the surface of the substrate outside the contact line) equal to the interfacial work of adhesion $W_{ad}$. The key difference is the inclusion of surface tension in the calculation of the energy release rate. The procedure is as follows. First, we obtain the "Hertz-like" solution of a contact problem with surface tension. By this we mean the relationship between the line force $P_H(a)$, indentation displacement $\delta_H(a)$, and contact width $2a$ in the absence of adhesive forces. From this solution, we compute the instantaneous contact compliance, defined by

$$C(a) = d\delta_H / dP_H = \frac{d\delta_H}{da} / \frac{dP_H}{da}. \tag{2}$$

As shown by Vajpayee *et al.*[39], knowing the instantaneous contact compliance and the Hertz-like load $P_H$ suffices to determine the energy release rate. Specifically, the energy release rate is given by[1]:

$$G = \frac{1}{4}\Lambda_H (P - P_H)^2, \qquad \Lambda_H = -\frac{dC}{da}. \tag{3}$$

Finally, the relation between contact width *2a* and the applied line load *P* is obtained using the energy balance condition $G = W_{ad}$. Using (3), this relation is

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[1] In Vajpayee et al. [39], $G = -\frac{1}{2}\frac{dC}{dA}(P - P_H)^2$, where A is the contact area. In our case, the area of contact is 2a since the length of the cylinder is infinite. Note P and $P_H$ has units of force per unit length.

$$P = P_H - 2\sqrt{W_{ad} / \Lambda_H} \ . \qquad (4)$$

It is important to note that in general, $dC/da$ and $P_H$ are functions of the shear modulus, surface tension, cylinder radius, and the contact width. Once these two functions are determined (see below), equation (4) generalizes the JKR theory to include surface tension.

Equations (3-4) apply equally well to the three-dimensional indentation problem for which they were first derived[39] and for the two-dimensional cylindrical problem analyzed here, which is obtained by assuming that the cylinder is infinitely long. There is, however, a peculiarity of two-dimensional contact problems. As noted by Johnson[22], the displacement of a point in an elastic half-space loaded by a line force (resulting in a two dimensional stress field) cannot be expressed relative to a datum at infinity, since the displacements decrease with distance *r* from the contact zone as ln *r*. Here we follow Johnson[22] by defining the displacements relative to an arbitrary point x = c >> a (see Fig. 1) on the surface of the substrate. Thus, the indenter displacement $\delta_H$ in (2) can take any value depending on the choice of this point. Nevertheless, as shown below, d*C/da* in equations (3-4) does not depend on the choice of *c*, as long as *c* >> *a*.

**2.1 The elasticity dominated limit.**

For an infinitely long cylinder ($L \to \infty$), in the limit where surface tension can be neglected and resistance to deformation is dominated by elasticity, the relation between contact width and load in the presence of adhesion has been derived earlier by Barquins [31] and Chaudhury *et al.*[32] using a different approach. Here, for uniformity of development, it is instructive to derive this relation using equation (4).

In the absence of surface tension and adhesion, the relationship between force and contact width is [22]

$$P_H = \frac{\pi \mu a^2}{R} \ . \qquad (5)$$

Because of the two-dimensional nature of the problem, and the attendant dependence of displacements on choice of datum referred to above, determining the instantaneous contact compliance require special consideration. The normal surface displacement v for $|x|$ > a can be obtained using a result given in Johnson[22]:

$$v(|x|>a)=\frac{1}{R}\left[\frac{|x|}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}ln\left||x|+\sqrt{x^2-a^2}\right|-\frac{x^2}{2}\right]+b\,, \tag{6}$$

where *b* is an arbitrary constant. We pick a point at x = c >> a as datum, i.e., we define the normal displacement at c to be zero, and this determines b. Continuity of v across the contact line implies that

$$\delta_H=v\left(x=a\right)+\frac{a^2}{2R}=-\frac{1}{R}\left[\frac{c}{2}\sqrt{c^2-a^2}-\frac{a^2}{2}ln\left|\frac{c+\sqrt{c^2-a^2}}{a}\right|-\frac{c^2}{2}\right]\approx\frac{a^2}{2R}ln\left(\frac{2c}{a}\right)+\frac{a^2}{4R}\quad c>>a. \tag{7}$$

The compliance, evaluated using (2), (5) and (7) is found to be:

$$C=\frac{1}{2\pi\mu}ln\left[2c/a\right]\Rightarrow dC/da=-\Lambda_H(a)=\frac{1}{2\pi\mu a}\,. \tag{8}$$

Substituting (5) and (8) into (4) recovers the JKR theory for a rigid cylinder in adhesive contact with an incompressible elastic half space [31] , i.e.,

$$P=\frac{\pi\mu a^2}{R}-2\sqrt{2\pi\mu a W_{ad}}\ . \tag{9a}$$

Note at zero load, the half contact width $a_0$ caused by adhesive forces is:

$$a_0=2\left(\frac{R^2W_{ad}}{\pi\mu}\right)^{1/3}. \tag{9b}$$

The pull-off load is found to be[22],

$$P_{off}=-3\pi\mu R^{1/3}\left(\frac{W_{ad}}{2\pi\mu}\right)^{2/3}. \tag{9c}$$

The contact half width at pull-off is:

$$a_{off} = R^{2/3}\left(\frac{W_{ad}}{2\pi\mu}\right)^{1/3}. \qquad (9d)$$

Recall that the pull-off load of a sphere is directly proportional to its radius and is independent of elastic modulus[23]. Equation (9c) shows that the scaling for pull-off is different for the case of cylinders.

## 2.2 The surface tension dominated limit.

Before tackling the general problem of the transition from the elasticity dominated limit to surface tension dominated limit, we examine the latter limiting case. The free body diagram in Fig. 2 shows that force balance requires

$$P = 2\sigma\sin\left(\theta - \theta_p\right), \qquad (10)$$

where $\theta = \sin^{-1}(a/R)$ and $\theta_p$ is the "peel" angle defined in Fig. 2.

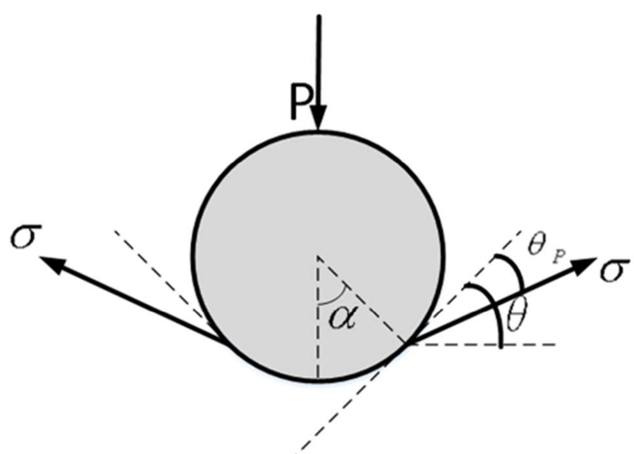



Fig. 2 Force balance and geometry in the surface tension limit.

The energy release rate $G$ can be computed using Kendall's peel theory [40]:

$$G = \sigma\left(1 - \cos\theta_p\right), \qquad (11)$$

Note, by setting $W_{ad} = \gamma_{SA} + \gamma_{CA} - \gamma_{CS}$ and use the fact that surface tension and surface energy has the same value for liquid (In general, this is not assumed in our calculation), the energy release-rate balance equation $G = W_{ad}$ is equivalent to Young-Dupre equation

$$\gamma_{CS} + \sigma\cos\left(\pi - \theta_p\right) = \gamma_{CA} \ , \tag{12}$$

where $\gamma$'s are the surface energies, the subscripts $C, S, A$ stand for rigid cylinder, compliant substrate and air, respectively. Setting $G = W_{ad}$ in (10) allows us to solve for $\theta_p$. Substituting $\theta_p$ into (9) gives:

$$\frac{P}{2\sigma} = \frac{a}{R}\left(1 - \frac{W_{ad}}{\sigma}\right) - \sqrt{1 - \frac{a^2}{R^2}}\sqrt{1 - \left(1 - \frac{W_{ad}}{\sigma}\right)^2} \ . \tag{13}$$

Equation (13) is valid for large deformation. In the small deformation limit, the angles must be small, and (13) reduces to a linear relation between load and contact width, i.e.,

$$P = \frac{2\sigma a}{R} - 2\sqrt{2\sigma W_{ad}} \ . \tag{14a}$$

The first term in (14a) can be identified as the Hertz-like load i.e., $(W_{ad} = 0)$,

$$P_H = 2\sigma a / R \ . \tag{14b}$$

which can also be found directly found using force balance (e.g. setting $\theta_p = 0$ in Fig. 2). Comparing (14a) and (4),

$$dC / da = -1 / 2\sigma \ . \tag{14c}$$

The half contact width at zero load $a_0$ is easily found by setting *P* = 0 in (14a),

$$a_0 = 2R\sqrt{\frac{W_{ad}}{2\sigma}} \tag{15}$$

Note that $a_0$ is directly proportional to the cylinder radius, whereas in the elasticity dominated limit, this width scales with radius to the 2/3 power. The pull-off load in this limit is independent of the radius of cylinder, i.e.,

$$P_{off} = -2\sqrt{2\sigma W_{ad}} \ , \tag{16}$$

in contrast with the elasticity dominated limit, where the pull-off force scales with $R^{1/3}$. In particular, in the surface tension dominated limit, the cylinder pulls off at $a = 0$ whereas in the elasticity dominated limit the cylinder jumps out of contact at a *finite* width $a_{pull-off}$ given by (9d).

## 3 Transition between the elasticity and surface-tension-dominated limits:

### 3.1 Formulation of the Hertz problem with surface tension

The transition between the elasticity and surface tension dominated limits is completely specified by the Hertz-load $P_H$ and the derivative of the contact compliance $\Lambda_H$. To determine these quantities, we solve the indentation Hertz problem with surface tension (but without adhesion). The geometry is shown in Fig. 1. Due to surface tension, the normal stress can be discontinuous across the surface. We denote by

$$\lim_{y\to 0^{\pm}} \tau_{yy} \equiv \tau_{yy}^{\pm}(x) \tag{17a}$$

where $\tau_{yy}^{+}(x)$ and $\tau_{yy}^{-}(x)$ are the stress normal to the surface as it is approached from inside and outside of the solid. Outside the contact zone $x \in [-a,a], y=0$, the normal surface traction approached from outside of the solid is zero, which is represented by

$$\lim_{y\to 0^{-}} \tau_{yy} \equiv \tau_{yy}^{-}(x) = 0, \ \ x \notin [-a,a], \tag{17b}$$

The difference between the normal stress just inside and just outside the solid is given by the Laplace equation, which enforces force balance,

$$\tau_{yy}^{-} - \tau_{yy}^{+} = \sigma v'' \qquad\qquad |x| < \infty \tag{17c}$$

where $v$ is the normal component of surface displacement and we have assumed that curvature of the surface can be approximated by its second derivative with respect to 'x'. That is, deformation of the surface outside the contact zone gives rise to a Laplace pressure. Therefore, even though the surface outside the contact zone is traction free, the normal component of the stress in the solid, $\tau_{yy}^{+}$, as one approaches the surface y = 0 from y > 0 is not zero.

In addition, the incompressibility and traction-free assumption implies that

$$\tau_{xy}(|x| < \infty, y = 0) = 0 \tag{17d}$$

The starting point of our analysis begins with a well-known result in elasticity (see Johnson[22]), that the gradient of vertical surface displacement v of an incompressible elastic half space is related to the normal traction $\tau_{yy}^{+}$ on the surface y = 0 by

$$\frac{dv}{dx}=\frac{1}{2\pi\mu}PV\int_{-\infty}^{\infty}\frac{\tau_{yy}^{+}(t)dt}{x-t} \quad , \tag{18}$$

where μ is the shear modulus of the substrate, PV stands for principal value integral and a prime denotes derivative with respect to x. As already mentioned, because of surface tension, even though the surface outside the contact zone is traction free, the normal component of the stress $\tau_{yy}^{+}$ as one approaches the surface y = 0 from y > 0 is not zero.

Using the Hilbert transform, we invert (18) obtaining:

$$\frac{1}{2\mu}\tau_{yy}^{+}(x)=-\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{dv(t)/dt}{x-t}dt \tag{19}$$

To simplify the analysis, we introduce the following normalization:

$$\bar{x}=x/a,\bar{t}=t/a \quad v=\bar{v}a^2/R\ ,\ s=\frac{\tau_{yy}^{+}(\bar{t})R}{4\mu a} \tag{20}$$

Using these normalized variables, (19) becomes:

$$s(\bar{x})=-\frac{1}{2\pi}PV\int_{-\infty}^{\infty}\frac{\bar{v}'(\bar{t})d\bar{t}}{\bar{x}-\bar{t}}\ , \tag{21}$$

where a prime denotes derivative with respect to $\bar{x}$. While the displacement outside the contact zone is unknown, the displacement inside the contact zone is given by the contact condition. In normalized form, this condition is:

$$\bar{v}'(|\bar{x}|<1)=\frac{d\bar{v}}{d\bar{x}}=-\bar{x} \tag{22}$$

Due to symmetry, $\sigma_{yy}^{+}$ and $\bar{v}'$ are even and odd function of $\bar{x}$ respectively. Using (22) and integrating by parts, after some calculations, (21) reduces to

$$s(\bar{x})=-\frac{1}{2\pi}\left[\int_{1}^{\infty}\bar{v}''(\bar{t})\ln\left|\bar{t}^2-\bar{x}^2\right|d\bar{t}+\left(2+\bar{x}\ln\left|\frac{\bar{x}-1}{\bar{x}+1}\right|-\ln\left|\bar{x}^2-1\right|\right)\right] \tag{23}$$

It should be noted that, in carrying out integration by parts, we enforced the continuity of $\bar{v}'$ across the contact line – ensuring bounded stress. The traction free boundary condition outside the contact zone requires $\tau_{yy}^{-}(|x|>a)=0$, so it follows from (17c) that $\tau_{yy}^{+}$ alone must balance the Laplace pressure induced by the surface curvature, $v''$:

$$-\tau_{yy}^{+}=\sigma v'' \Rightarrow s=-\beta\bar{v}'' , \qquad |\bar{x}|>1 \tag{24a}$$

where

$$\beta \equiv \frac{\sigma}{4\mu a}, \tag{24b}$$

is the ratio of the elasto-capillary length $\sigma/\mu$ and the contact width. Substituting (24a) in (23) gives the integral equation for the unknown vertical displacement outside the contact zone:

$$2\pi\beta\bar{v}''(\bar{x})=\int_{1}^{\infty}\bar{v}''(\bar{x})\ln\left|\bar{t}^{2}-\bar{x}^{2}\right|d\bar{t}+\left(2+\bar{x}\ln\left|\frac{\bar{x}-1}{\bar{x}+1}\right|-\ln\left|\bar{x}^{2}-1\right|\right), \quad |\bar{x}|>1 \tag{25}$$

Equation (25) implies that $\bar{v}''$ depends on *the single dimensionless parameter* $\beta$ introduced in (24b). Once v" is found outside the contact zone, we use (23) and (20) to determine the normal stress $\tau_{yy}^{+}$ inside the contact zone. Finally, it can be shown that, in the elasticity dominated limit where $\beta=0$, the exact solution of (25) is identical to the classical Hertz solution given by (6).

It should be noted that our formulation is exact within the assumption of small strain and linear elasticity. Because the Green's function of the line load problem does not have a close form solution, we do not use this approach. Instead, in our formulation the surface tension appears as a boundary condition, as given by (17c). This equation is coupled to (19), which relates the deformation of the free surface to the normal stress distribution. Note that (19) uses the conventional Green's function of the 2D elasticity problem. In our method, we could recover the Green's function of the surface tension problem by solving the integral equation (23) numerically. It can be shown that this Green's function decays logarithmically at distances *far* from the line of application of the line force and this will lead to a divergence of the displacement field at infinity, as stated in the last paragraph before section 2.1.

### 3.2 Determination of $P_H$ and $\Lambda_H = -dC/da$

The extended JKR theory, (4), can be written in the normalized form:

$$\hat{P} = \frac{\overline{P}_H \hat{a}^2}{3} - \frac{4}{3}\sqrt{\hat{a}/\overline{\Lambda}_H} \tag{26}$$

where

$$\hat{P} = \frac{P}{3\pi\mu R^{1/3}\left(\frac{W_{ad}}{2\pi\mu}\right)^{2/3}},\ \overline{P}_H \equiv \frac{R}{\pi\mu a^2}P_H,\ \hat{a} = \frac{a}{R^{2/3}\left(\frac{W_{ad}}{2\pi\mu}\right)^{1/3}},\ \overline{\Lambda}_H = 2\pi\mu a\Lambda_H, \tag{27}$$

are the normalized indentation force, Hertz load with surface tension, half contact width and derivative of compliance respectively. Note that indentation load $P$ and half contact width $a$ are normalized by the magnitude of pull-off load and pull-off radius in elasticity dominated limit. In the absence of adhesion, $P_H$ must be balanced entirely by $\tau_{yy}^{-}$. Using (17c) and (22), this condition is

$$P_H = -\int_{-a}^{a}\tau_{yy}^{-}(x)dx = -\int_{-a}^{a}\tau_{yy}^{+}(x)dx + \frac{2\sigma a}{R} \Rightarrow \overline{P}_H = \frac{4}{\pi}\left[2\beta - \int_{-1}^{1}s(\overline{x},\beta)d\overline{x}\right]. \tag{28}$$

*The last equality in (28) shows that* $\overline{P}_H$ *is a function* of $\beta$ *only*. Since $\overline{P}_H(\beta=0)=1$, this normalization measures the deviation of the Hertz-like load from the elastic limit. It is evaluated by solving (25) for $\overline{v}''$, then use (23) to evaluate $s(\overline{x},\beta)$ for $|\overline{x}|<1$. The numerical method we used to solve the integral equation (25) is given in the supplementary material; here we state the relevant results. As shown in Fig. 3, $\overline{P}_H$ can be well approximated by the expression:

$$\overline{P}_H = 1 + \left(\frac{8\beta}{\pi}\right)\left(\frac{\beta^2 + 1.819\beta + 0.1146}{\beta^2 + 1.164\beta + 0.03607}\right) \tag{29}$$

It should be noted that (29) gives the correct asymptotic behaviors for small and large $\beta$, which correspond to the elasticity and surface tension dominated limits, respectively.

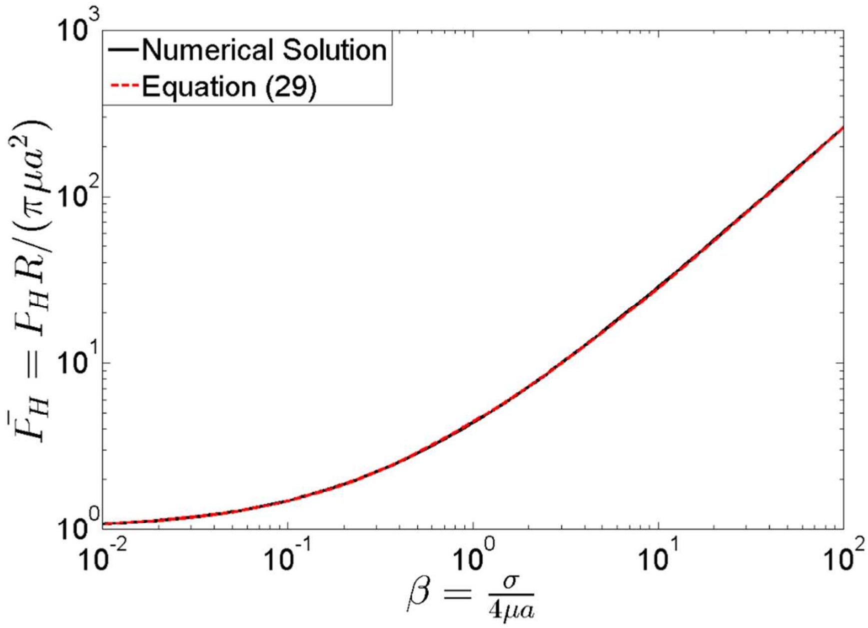



Fig. 3 Normalized Hertz-like load $\overline{P}_H$ versus normalized surface tension $\beta$. In this plot, the numerical solution is indistinguishable from (29). This result matches well with the results of Long *et al.*[33], and here a fitted formula is provided.

To expedite the analysis, it is convenient to define a normalized indentation depth $\overline{\delta}_H$ as

$$\overline{\delta}_H \equiv \delta_H R / a^2 \ . \tag{30}$$

Same as in the elasticity limit, we fixed a point *c* at a distance much greater than the contact width, and define the displacement there to be zero; as a result,

$$\begin{aligned}\overline{\delta}_H &= \overline{v}\big|_{\overline{x}=0} - \overline{v}\big|_{\overline{x}=\overline{c}} = \left(\overline{v}\big|_{\overline{x}=0} - \overline{v}\big|_{\overline{x}=1}\right) + \left(\overline{v}\big|_{\overline{x}=1} - \overline{v}\big|_{\overline{x}=\overline{c}}\right) \\ &= \frac{1}{2} - \int_1^{\overline{c}} \overline{v}'(\overline{x})d\overline{x} = \frac{1}{2} + \int_1^{\overline{c}} \left[1 - \int_1^{\overline{x}} \overline{v}''(\overline{t},\beta)d\overline{t}\right] d\overline{x} = \overline{c} - \frac{1}{2} - \int_1^{\overline{c}} \int_1^{\overline{x}} \overline{v}''(\overline{t},\beta)d\overline{t}\,d\overline{x}\end{aligned} \tag{31}$$

where $\overline{c} = c / a$. Equation (31) allows us to determine $\overline{\delta}_H$ using the solution of the integral equation (25). However, unlike $\overline{P}_H$, which depends only on $\beta$, $\overline{\delta}_H$ is a function of both $\overline{c}$ and $\beta$; thus the normalized instantaneous contact compliance in general depends on $\overline{c}$ *and* $\beta$, even though $\overline{\Lambda}_H$ is independent of $\overline{c}$. In the supplementary material, we show that $\overline{\Lambda}_H$ can be fitted by

$$\bar{\Lambda}_H(\beta) = \frac{\pi\beta^2 + 9.102\beta + 1}{4\beta^3 + 20.82\beta^2 + 12.75\beta + 1}. \quad (32)$$

It should be noted that (32) gives the correct asymptotic behavior for small and large $\beta$.

### 3.3 Extended JKR theory with surface tension

Equations (26), (29) and (32) completely specify the relation between applied load (indentation force) and the contact width. We call this relation the extended JKR theory; it is the main result of this work. Although $\bar{\Lambda}_H$ and $\bar{P}_H$ are functions only of $\beta$, the contact width is normally an unknown, so it is convenient to define a different dimensionless quantity $\omega$,

$$\omega = \beta\hat{a} = \frac{\sigma}{4(\mu R)^{2/3}(W_{ad}/2\pi)^{1/3}}. \quad (33)$$

Since $\bar{P}_H$ and $\bar{\Lambda}_H$ in (26) depend only on $\beta = \omega/\hat{a}$, (26) states that the extended JKR theory ($\hat{P} - \hat{a}$ relation) is controlled by a *single* dimensionless parameter $\omega$ which is consistent with the work of Cao *et al*.[30]. Fig. 4 plots the normalized contact radius $\hat{a}$ versus the normalized load $\hat{P}$ for different $\omega$ using (26), (29) and (32). The limiting case of $\omega = 0$ (JKR, elastic limit) is plotted in the same figure as a reference.

**Pull-off load and radius**

For the elastic limit, it is well known that in a load controlled test, the cylinder suddenly jumps off the substrate at a finite contact width given by (9d) with pull-off load given by (9c). Fig. 4 shows that, the contact width at pull-off decreases with increasing ω; and eventually vanishes at a critical value at $\omega \approx 1.6$. From Fig. 4, it is shown that $\left.\frac{d\hat{P}}{d\hat{a}}\right|_{\hat{a}=0} = 0$ at this critical value. From (26) and use the fact that $\frac{d\beta}{d\hat{a}} = \frac{d(\omega/\hat{a})}{d\hat{a}} = -\frac{\omega}{\hat{a}^2}$,

$$\left.\frac{d\hat{P}}{d\hat{a}}\right|_{\hat{a}=0} = \frac{8\omega}{3\pi} - \frac{2}{3}\sqrt{\frac{1}{\hat{a}\bar{\Lambda}_H}}\left(1 + \frac{\beta}{\bar{\Lambda}_H}\frac{d\bar{\Lambda}_H}{d\beta}\right) \quad (34a)$$

Since $\beta = \omega/\hat{a}$, $\beta \to \infty$ when $\hat{a} \to 0$. Using (32), we found

$$\frac{\beta}{\bar{\Lambda}_H}\frac{d\bar{\Lambda}_H}{d\beta} = -1 + \frac{2.30774}{\beta} + o(\frac{1}{\beta^2}) \ . \quad (34b)$$

Substituting this relationship into (34a) gives

$$\left.\frac{d\hat{P}}{d\bar{a}}\right|_{\bar{a}=0} = \frac{8\omega}{3\pi} - 1.736\sqrt{\frac{1}{\omega}} \quad (35)$$

Setting (35) equal to zero gives the critical value of $\omega$, $\omega_c \approx 1.6$.

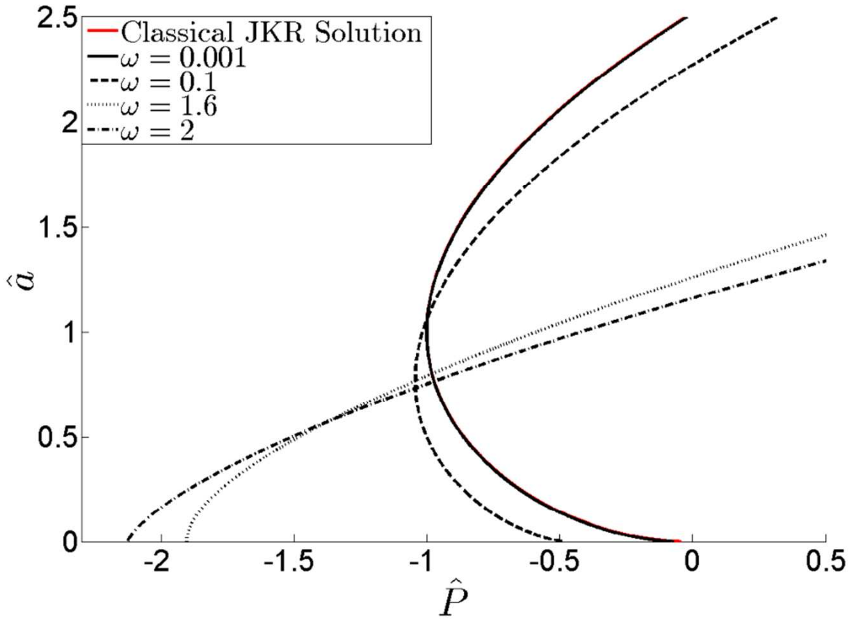
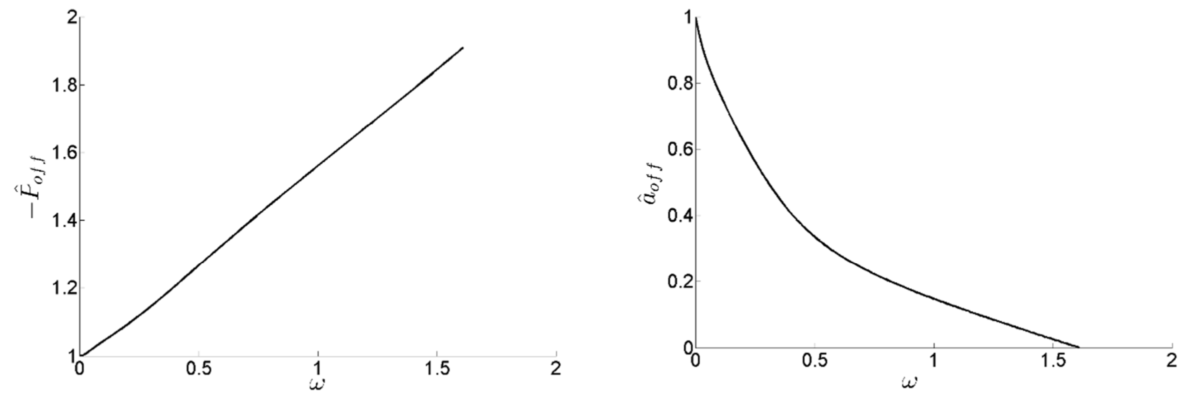



Fig. 4 Normalized contact radius versus normalized indentation force for different values of $\omega$.

Thus, our theory predicts that for $\omega > \omega_c$, the cylinder pulls off at zero contact width from the substrate.

Fig. 5 highlights the surface tension effect on pull-off load and pull-off width and the way we normalize the load and radius reflect the deviation of pull-off behavior from the elasticity dominated limit (classical JKR solution). It should be noted that under this normalization, both pull-off load and pull-off width depend only on a single dimensionless parameter $\omega$ defined by (33).




(a) (b)

Fig. 5 (a) Normalized pull-off load versus $\omega$ (b) Normalized contact half width versus $\omega$.

## Summary and Discussion

In this paper we analyzed the adhesive contact mechanics of a rigid cylinder indenting an elastic half-space, extending previous analyses to account for the influence of surface tension. Specifically, we show that the classical JKR theory without surface tension can be extended to include surface tension by modifying the Hertz load and the compliance. By introducing the normalization in (27), we manage to collapse the classical JKR solution (9a) and the liquid-like limit solution (14a) into a single relation (26) which is completely specified by a single dimensionless parameter $\omega$. Physically, $\omega$ is essentially the ratio of the elasto-capillary length $\sigma/\mu$ to a characteristic length $R^{2/3}\left(W_{ad}/\mu\right)^{1/3}$ where adhesion acts (the pull-off width).

There is a significant difference between the pull-off mechanics of a rigid sphere and a rigid long cylinder in contact with a soft elastic substrate. For a sphere with radius $R_s$, Hui *et al.*[19] has recently shown that the pull-off load is found to vary between $-3\pi R_s W_{ad}/2$ and $-\pi R_s W_{ad}$ for the elasticity dominated and surface tension dominated limit respectively. Note that the pull-off force in both limit is directly proportional to the radius of the sphere and the work of adhesion. In particular, pull-off always occur at a finite contact radius. This is not the case for a rigid cylinder, where the pull-off force is directly proportional to the square root of the product of the surface tension and the work of adhesion, and is independent of the cylinder radius in the liquid limit. In addition, the contact radius at pull-off vanishes above a critical value of $\omega$, $\omega_c \approx 1.6$, implying that surface tension could strongly affect pull-off behavior for long cylinder objects. The deformed shape in the surface-tension-dominated limit also depends on whether the contactor is a sphere or a long cylinder. For example, Carrillo *et al*[20] found that it is necessary to assume the particle deforms into a spherical cap plus a cylindrical neck in order for their model to agree with their molecular dynamics simulation in the surface tension dominated limit. The deformed shape of our substrate do not looked like a neck, despite the fact that our analysis does contain the "necking" limit, by which we mean the limit in which surface tension dominates. An important difference is that our analysis is in two-dimensions whereas that of Carrillo *et al*[20] is for a deformable axisymmetric spherical particle. Therefore, in the absence of gravity, the shape of the surface near the cylinder must be planar (which does not "look" like a neck) in the surface-tension-dominated limit. Finally, we are not aware of any experiments on contact mechanics of cylinders on substrates where surface tension effects are important. It will be very interesting to see whether our prediction is consistent with future experimental observations.

There are obvious limitations in our model, we use small strain theory where the contact radius is assumed to be small in comparison with the radius of the cylinder. This condition is not expected to hold in experiments involving very soft substrates. For a rigid sphere in contact with soft substrates, it has been found that the small strain theory is amazingly good for contact radius as large as 75% of the sphere radius (in liquid limit $W_{ad} << \sigma$ is assumed)[19]. Of course, the same conclusion may not apply for the cylindrical geometry. In a future work, we will use large deformation theory to quantity the limitations of the small strain theory. Also, our calculation is strictly valid for no-slip contact. For substrates such as hydrogels, the frictionless boundary condition could be more appropriate. Recall that the classical JKR theory is strictly valid for frictionless and no slip contact, as long as the substrate is incompressible. However, as pointed out in section 2, without the no-slip condition, one would have to introduce new physics into the model since the tension of the substrate surface will change once it is in contact with the rigid surface. The problem is that the interfacial tension between a rigid surface and the surface of a compliant solid is not well defined, as a rigid surface can support any tension. From the mechanics point of view, there is no consistent way to assign the tension acting on the substrate surface after contact. One way to deal with this difficulty is to assign the same tension to the substrate surface after contact, which is essentially the approach of Long *et a.l*[33]. In this case the results in this paper apply without modification. However, the effect of this approximation could result in a work of adhesion that is different from that given by the Young-Dupre equation. Whether this ad-hoc assumption is justified will depend on future experimental data.

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**Abstract**

This article studies how surface tension affects the adhesive contact mechanics of a long rigid cylinder on an infinite half space comprising of an incompressible elastic material. We present an exact solution based on small strain theory. The relation between indentation force and contact width is found to depend on a single dimensionless parameter $\omega = \frac{\sigma}{4(\mu R)^{2/3}(W_{ad}/2\pi)^{1/3}}$ , where $R$ is the cylinder radius, $W_{ad}$ the interfacial work of adhesion, $\sigma$ and $\mu$ are the surface tension and shear modulus of the half space respectively. The solution for small $\omega$ reduces to the classical Johnson-Kendall-Roberts (JKR) theory whereas for large $\omega$, the solution reduces to the small slope version of the Young-Dupre equation. The pull-off phenomenon is carefully examined and it is found that the contact width at pull-off is reduced to zero when surface tension is larger than a critical value.