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## Pure measures of bending for soft plates

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This paper, originally motivated by a question raised by Wood and Hanna [*Soft Matter*, 2019, **15**, 2411], shows that pure measures of bending for soft plates can be defined by introducing the class of bending-neutral deformations, which represent finite incremental changes in the plate's shape that do not induce any additional bending. This class of deformations is subject to a geometric compatibility condition, which is fully characterized. A tensorial pure measure of bending, which is invariant under bending-neutral deformations, is described in detail. As shown by an illustrative class of examples, the general notion of a pure measure of bending could be useful in formulating direct theories for soft plates, where stretching and bending energies are treated separately.

### 1 Introduction

It has been remarked that a pure measure of bending for extensible rods cannot be provided by the curvature of its centerline, as this can easily be shown to be affected by a superimposed dilation.<sup>1</sup> Actually, within the nonlinear theory of extensible elastic rods, this issue has long been known, at least since the work of Antman<sup>2</sup> and others who soon followed in his footsteps.<sup>3,4</sup> It also has consequences on the choice of the appropriate form of the bending component of the strain energy, when in the direct approach one wishes to separate it neatly from the stretching component.

This was indeed the criterion advocated in ref. 2 for a planar extensible *elastica*: if  $\vartheta$  denotes the centerline's deflection angle (relative to a fixed direction in the plane) of a naturally straight rod, the appropriate quadratic bending energy independent of stretching would be proportional to  $(\partial_x \vartheta)^2$ , and not  $(\partial_s \vartheta)^2$ , where  $x$  and  $s$  denote the arc-length coordinate in the reference and present configurations, respectively. As also remarked in ref. 2, a theory measuring the bending energy of an extensible rod by  $(\partial_s \vartheta)^2$ , such as that proposed in ref. 5, would generally be more complicated.†

Other attempts to derive the balance equations for an extensible elastic rod have been based on Biot's *nominal* strain tensor<sup>8</sup> as proposed in ref. 6 and 7 (also see ref. 9 and 10); they corroborated the approach purported in ref. 2. A similar corroboration was provided by ref. 11, which followed yet another avenue, designed according to the canons of modern continuum mechanics.

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† Indeed the major advantage of Antman's proposed constitutive law seems to be its simplicity, as witnessed by the neat analysis that ensued in ref. 2.

Similarly, for a plate whose planar midsurface  $S$  in the reference configuration gets deformed into the surface  $\mathcal{S}$  embedded in three-dimensional space, the invariants of the curvature tensor  $\nabla_s \nu$ , where  $\nu$  is the outer unit normal to  $\mathcal{S}$  and  $\nabla_s$  denotes the surface gradient, cannot serve as pure measures of bending, as also neatly illustrated in ref. 12.

In analogy with the theory of stretchable rods, proposals for the bending energy of plates and shells (*i.e.*, plates curved in their natural state) have been put forward that are not affected by one or another form of stretching. The approaches taken to achieve this goal are disparate, as are the conclusions reached; controversies abound, especially for shells, which I deliberately leave out of the scope of this paper.

In an alternative approach to geometric elasticity as formulated in ref. 13, a theory for deformable, geometrically incompatible sheets was derived in ref. 7 from a three-dimensional model phrased in terms of Biot's strain measures. The outcome differs from that in ref. 13, also in the rod-limit case, where the alternative energy is closer to Antman's. The theory of ref. 7 has been applied to a soft matter system in ref. 14; it effectively employs a measure of bending for plates that has a distant antecedent in ref. 15.

More recently, a dimension reduction method following the general lines proposed in ref. 16–19 was applied in ref. 20 to a three-dimensional isotropic quadratic strain energy formulated in terms of Biot's strain tensor to justify a model for shells.<sup>21</sup> In particular, in the limiting case of plates, the one that concerns us here, two two-dimensional bending measures arose naturally from the method employed in ref. 20, one of these measures remains invariant under arbitrary superimposed stretchings, and the other remains invariant only under superimposed dilations, both reducing to Antman's measure in the rod-limit.

The questions addressed in this paper will be of a pure kinematic nature. First, we need to identify a criterion of *bending-neutrality* for soft (highly deformable) plates: we must



single out the incremental deformations that change the shape of  $\mathcal{S}$ , but do not introduce any further bending. Thus, *pure* measures of bending will be defined as the ones *invariant* under bending-neutral deformations. I shall focus on one isotropic such measure,  $\mathbf{A}$ , tensorial in character, alongside with its scalar invariants and contrast them with both similar and dissimilar measures from the literature. It will be shown that  $\mathbf{A}$  reduces to Antman's measure for extensible rods.

Identifying the pure measure of bending appropriate for a class of materials should prelude to a proposal for a direct theory of plates for which stretching and bending modes are kept separate from one another.

This paper is organized as follows. In Section 2, we define the class of bending-neutral deformations and present an isotropic measure of pure bending associated with them. Section 3 is devoted to the compatibility condition that a bending-neutral deformation must obey; we shall see that this class contains both pure tangential stretching deformations and planar *drill* rotations, characterized separately in ref. 22 and 23. Finally, in Section 4, we summarize the conclusions of this study and comment briefly about its possible extensions.

## 2 Bending-neutral deformations

Consider a surface  $S$  lying on the  $(x_1, x_2)$  plane of a Cartesian frame  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ , so that  $\mathbf{e}_3$  is one of its unit normals. Let  $\mathbf{x}$  denote the position vector in  $S$  and let  $\mathbf{y}$  be the deformation that maps  $S$  onto  $\mathcal{S}$  in three-dimensional Euclidean space  $\mathcal{E}$ . We shall assume that  $\mathbf{y}$  is at least of class  $C^2$  and we shall denote by  $\boldsymbol{\nu}$  the unit normal to  $\mathcal{S}$  oriented coherently with the orientation of  $\mathbf{e}_3$  (see Fig. 1).

Extending to the present geometric setting results well-known from three-dimensional kinematics (see, e.g., ch. 6 of ref. 24), the deformation gradient  $\nabla\mathbf{y}$  can be represented as

$$\nabla\mathbf{y} = \lambda_1 \mathbf{l}_1 \otimes \mathbf{r}_1 + \lambda_2 \mathbf{l}_2 \otimes \mathbf{r}_2, \quad (1)$$

where  $\nabla$  denotes the gradient in  $\mathbf{x}$ , the positive scalars  $\lambda_1$  and  $\lambda_2$  are the *principal stretches* and the unit vectors  $\mathbf{r}_i, \mathbf{l}_i$  are the corresponding *right* and *left* principal directions of stretching. While  $\mathbf{r}_1(\mathbf{x})$  and  $\mathbf{r}_2(\mathbf{x})$  exist on the  $(x_1, x_2)$  plane for all  $\mathbf{x} \in S$ ,  $\mathbf{l}_1(\mathbf{x})$  and  $\mathbf{l}_2(\mathbf{x})$  exist on the tangent plane to  $\mathcal{S}$  at  $\mathbf{y}(\mathbf{x})$ . The right and left Cauchy–Green tensors are correspondingly defined and represented by

$$\mathbf{C} := (\nabla\mathbf{y})^T (\nabla\mathbf{y}) = \lambda_1^2 \mathbf{r}_1 \otimes \mathbf{r}_1 + \lambda_2^2 \mathbf{r}_2 \otimes \mathbf{r}_2, \quad (2a)$$

$$\mathbf{B} := (\nabla\mathbf{y})(\nabla\mathbf{y})^T = \lambda_1^2 \mathbf{l}_1 \otimes \mathbf{l}_1 + \lambda_2^2 \mathbf{l}_2 \otimes \mathbf{l}_2. \quad (2b)$$

I wish to introduce a *bending-neutral* deformation as an incremental (finite) deformation  $\mathbf{y}'$  that maps  $\mathcal{S}$  into a surface  $\mathcal{S}^*$  without further bending it. To make this notion precise, we first consider the composed deformation  $\mathbf{y}^* := \mathbf{y}' \circ \mathbf{y}$ , where  $\mathbf{y}'$  is a general, smooth (at least  $C^2$ ) mapping that changes  $\mathcal{S}$  into  $\mathcal{S}^*$  and is characterized by having

$$\nabla\mathbf{y}' = \mathbf{R}'\mathbf{U}', \quad (3)$$

where  $\mathbf{R}'$  belongs to the appropriate orthogonal group in three dimensions,  $\text{SO}(3)$ , and

$$\mathbf{U}' = \lambda'_1 \mathbf{u}'_1 \otimes \mathbf{u}'_1 + \lambda'_2 \mathbf{u}'_2 \otimes \mathbf{u}'_2 \quad (4)$$

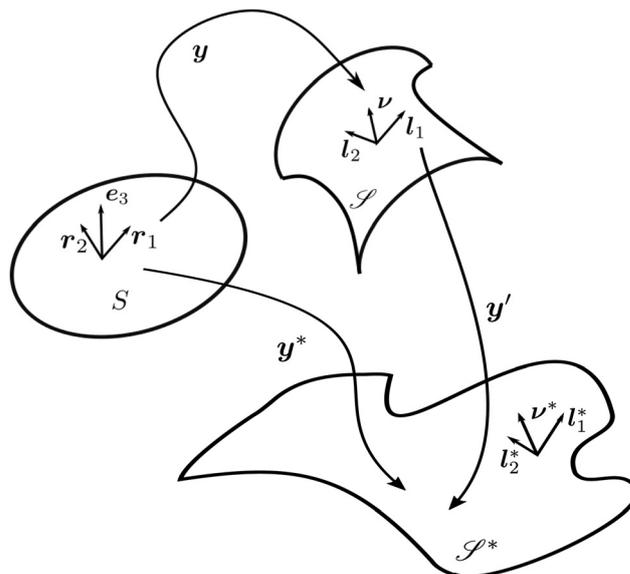


Fig. 1 A flat surface  $S$  in the  $(x_1, x_2)$  plane of a fixed Cartesian frame  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  is deformed by the mapping  $\mathbf{y}$  into a smooth surface  $\mathcal{S}$  embedded in three-dimensional Euclidean space  $\mathcal{E}$ . The unit vectors  $(\mathbf{r}_1, \mathbf{r}_2)$  are the right principal directions of stretching in the reference configuration, while  $(\mathbf{l}_1, \mathbf{l}_2)$  are the left principal directions of stretching in the present configuration;  $\mathbf{e}_3$  is the outer unit normal to  $S$ , while  $\boldsymbol{\nu}$  is the outer unit normal to  $\mathcal{S}$ ; they are oriented so that  $\mathbf{e}_3 = \mathbf{r}_1 \times \mathbf{r}_2$  and  $\boldsymbol{\nu} = \mathbf{l}_1 \times \mathbf{l}_2$ , respectively. Surface  $\mathcal{S}^*$  is obtained by combining  $\mathbf{y}$  with the incremental deformation  $\mathbf{y}'$ , so that  $\mathbf{y}^* = \mathbf{y}' \circ \mathbf{y}$ . The outer unit normal to  $\mathcal{S}^*$  is  $\boldsymbol{\nu}^*$  and the left principal directions of stretching associated with  $\mathbf{y}^*$  are  $\mathbf{l}_1^*$  and  $\mathbf{l}_2^*$ , with  $\boldsymbol{\nu}^* = \mathbf{l}_1^* \times \mathbf{l}_2^*$ .

is a *stretching* tensor with principal directions of stretching  $\mathbf{u}'_1, \mathbf{u}'_2$ , tangent to  $\mathcal{S}$  at  $\mathbf{y}(\mathbf{x})$  and principal stretches  $\lambda'_1, \lambda'_2 > 0$ . The tensor  $\mathbf{U}'$  is regarded as a (linear) mapping of the tangent plane to  $\mathcal{S}$  at  $\mathbf{y}(\mathbf{x})$  into itself.

Eqn (3) is the form of polar decomposition fit to describe the deformation of material surfaces; it was established in ref. 25 within the general coordinate-free theory introduced in ref. 26, which will also be followed here (see also ref. 27). A tensor  $\mathbf{V}'$  that maps the tangent plane to  $\mathcal{S}^*$  at  $\mathbf{y}^*(\mathbf{x}) = \mathbf{y}'(\mathbf{y}(\mathbf{x}))$  into itself, can also be introduced *via* the following equation,

$$\mathbf{R}'\mathbf{U}' = \mathbf{V}'\mathbf{R}'. \quad (5)$$

It follows from (5) that  $\mathbf{V}'$  possesses the same eigenvalues  $\lambda'_1, \lambda'_2$  as  $\mathbf{U}'$ , while its eigenvectors are  $\mathbf{R}'\mathbf{u}'_1, \mathbf{R}'\mathbf{u}'_2$ , which generally differ from the left principal directions of stretching  $\mathbf{l}_1^*, \mathbf{l}_2^*$  associated with  $\mathbf{y}^*$  (and shown Fig. 1). Using the chain rule,

$$\nabla\mathbf{y}^*(\mathbf{x}) = \nabla\mathbf{y}'(\mathbf{y}(\mathbf{x}))\nabla\mathbf{y}(\mathbf{x}) \quad \forall \mathbf{x} \in S, \quad (6)$$

(3) and (5) imply that

$$\mathbf{B}^* := (\nabla\mathbf{y}^*)(\nabla\mathbf{y}^*)^T = \mathbf{V}'(\mathbf{R}'\mathbf{B}\mathbf{R}'^T)\mathbf{V}', \quad (7)$$

which shows that the eigenframes of  $\mathbf{B}^*$  and  $\mathbf{V}'$  coincide whenever  $\mathbf{B}$  and  $\mathbf{U}'$  commute.‡

It should also be noted that since  $\mathbf{y}'$  is defined on  $\mathcal{S}$ , its gradient  $\nabla\mathbf{y}'$  featuring in (3) should be properly regarded as a

‡ This follows easily from applying both sides of (7) to the eigenvectors  $\mathbf{R}'\mathbf{u}'_i$  of  $\mathbf{V}'$ .



surface gradient on  $\mathcal{S}$ ,  $\nabla_s \mathbf{y}'$ , according to the definition given in ref. 26 (see also ref. 28).

We say that  $\mathbf{y}'$  is a *bending-neutral* deformation if

$$\mathbf{R}' = \mathbf{R}_0 \mathbf{R}_\nu, \quad (8)$$

where  $\mathbf{R}_0 \in \text{SO}(3)$  is a *uniform* rotation, *i.e.*, independent of position, and  $\mathbf{R}_\nu \in \text{SO}(\nu)$ , where  $\text{SO}(\nu)$  is the group of rotations about the unit normal  $\nu$ . While  $\mathbf{R}_0$  is a global rigid motion,  $\mathbf{R}_\nu$  is a local *drill* rotation. For  $\mathbf{R}'$  as in (8), (3) represents an incremental deformation that entails local stretching of  $\mathcal{S}$ , described by  $\mathbf{U}'$ , possibly followed by a *twist* distortion about  $\nu$ , described by  $\mathbf{R}_\nu$ , and an overall undistorting rotation  $\mathbf{R}_0$ .

We shall discuss in Section 3 the compatibility condition that a bending-neutral deformation  $\mathbf{y}'$  must obey to exist, at least locally, on  $\mathcal{S}$ . Here, we only consider that this class of incremental deformations includes both pure stretching deformations, for which  $\mathbf{R}_\nu$  is the identity, and pure drill rotations, for which  $\mathbf{U}'$  is the projection  $\mathbf{P}(\nu) := \mathbf{I} - \nu \otimes \nu$  on the plane orthogonal to  $\nu$ .<sup>§</sup> As shown in ref. 22 and 23, respectively, neither of these subclasses is empty.

## 2.1 Pure measures of bending

We say that a deformation measure  $\mathbb{A}(\mathbf{y})$ , regardless of its nature (scalar or tensor), is a *pure* measure of bending if it obeys the following invariance condition,

$$\mathbb{A}(\mathbf{y}^*) = \mathbb{A}(\mathbf{y}), \quad (9)$$

for all deformations  $\mathbf{y}^* := \mathbf{y}' \circ \mathbf{y}$ , such that  $\mathbf{y}'$  is bending-neutral.<sup>¶</sup>

Since bending is intuitively associated with curvature, to flesh out this definition we first need to determine how  $\nu^*$ , the outer unit normal to  $\mathcal{S}^*$ , is related to  $\nu$  when  $\mathbf{y}'$  is a bending-neutral deformation (and then also how  $\nabla \nu^*$  is related to  $\nabla \nu$ ). To this end, we first recall that

$$\mathbf{v}(\mathbf{x}) = \frac{(\nabla \mathbf{y})e_1 \times (\nabla \mathbf{y})e_2}{|(\nabla \mathbf{y})e_1 \times (\nabla \mathbf{y})e_2|} = \frac{\text{co}(\nabla \mathbf{y})e_3}{|\text{co}(\nabla \mathbf{y})e_3|}, \quad (10)$$

where  $\text{co}(\cdot)$  denotes the cofactor of a tensor (according to the definition in Section 2.11 of ref. 24). Since for any two tensors,  $\mathbf{L}$  and  $\mathbf{M}$ ,  $\text{co}(\mathbf{LM}) = \text{co}(\mathbf{L})\text{co}(\mathbf{M})$ , by (6) and (10), we have that<sup>||</sup>

$$\mathbf{v}^* = \frac{\text{co}(\nabla \mathbf{y}^*)e_3}{|\text{co}(\nabla \mathbf{y}^*)e_3|} = \frac{\text{co}(\nabla \mathbf{y}')\mathbf{v}}{|\text{co}(\nabla \mathbf{y}')\mathbf{v}|} = \mathbf{R}' \frac{\text{co}(\mathbf{U}')\mathbf{v}}{|\text{co}(\mathbf{U}')\mathbf{v}|}. \quad (11)$$

Now, if  $\mathbf{y}'$  is a bending-neutral deformation, since from (4)  $\text{co}(\mathbf{U}') = \lambda'_1 \lambda'_2 \mathbf{v} \otimes \mathbf{v} = (\det \mathbf{U}')\mathbf{v} \otimes \mathbf{v}$  and  $\mathbf{R}_\nu \nu = \nu$ , we learn from (11) and (8) that

$$\nu^* = \mathbf{R}_0 \nu. \quad (12)$$

Moreover, since  $\nabla \mathbf{R}_0 = 0$ , we see that

$$\nabla \nu^* = \mathbf{R}_0 \nabla \nu. \quad (13)$$

Thus, if we set

<sup>§</sup>  $\mathbf{I}$  represents the identity tensor in three space dimensions.

<sup>¶</sup> A different definition for a natural measure of bending was proposed in ref. 29 in the case of infinitesimal incremental deformations of  $\mathcal{S}$ .

<sup>||</sup> Use is also made here of the fact that  $\text{co}(\mathbf{R}) = \mathbf{R}$ , for any rotation  $\mathbf{R} \in \text{SO}(3)$ .

$$\mathbf{A}(\mathbf{y}) := (\nabla \nu)^T (\nabla \nu), \quad (14)$$

we readily see from (13) that this is a pure (tensorial) measure of bending, as

$$\mathbf{A}(\mathbf{y}^*) = (\nabla \nu^*)^T (\nabla \nu^*) = (\nabla \nu)^T (\nabla \nu) = \mathbf{A}(\mathbf{y}). \quad (15)$$

In the rest of the section, we shall focus on this measure. As defined in (14),  $\mathbf{A}$  is a linear mapping of the plane hosting the reference configuration  $S$  into itself, precisely as  $\mathbf{C}$  in (2a). Similar to the latter tensor,  $\mathbf{A}$  is both symmetric and positive definite. According to our definition, its scalar invariants are pure measures of bending as well.

It is perhaps worth noting that the tensor  $\mathbf{Z}(\mathbf{y}) := (\nabla \nu)(\nabla \nu)^T$ , similar in structure to  $\mathbf{B}$ , which maps the plane tangent to  $\mathcal{S}$  into itself, is *not* a pure measure of bending, although it has the same invariants as  $\mathbf{A}$ :  $\mathbf{Z}$  transforms as follows,  $\mathbf{Z}(\mathbf{y}^*) = \mathbf{R}_0 \mathbf{Z}(\mathbf{y}) \mathbf{R}_0^T$ .

We denote by the *curvature* tensor of  $\mathcal{S}$  by  $\nabla_s \nu$ , for which we adopt a sign convention opposite to the one customary in differential geometry.<sup>\*\*</sup>  $\nabla_s \nu$  is a symmetric tensor mapping the tangent plane to  $\mathcal{S}$  into itself; in its eigenframe, it is represented as

$$\nabla_s \nu = \kappa_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \kappa_2 \mathbf{n}_2 \otimes \mathbf{n}_2, \quad (16)$$

where  $\mathbf{n}_1, \mathbf{n}_2$  are the principal directions of curvature of  $\mathcal{S}$  and  $\kappa_1, \kappa_2$  are the corresponding principal curvatures.

Using the chain rule,

$$\nabla \nu(\mathbf{x}) = \nabla_s \nu(\mathbf{y}(\mathbf{x})) \nabla \mathbf{y}(\mathbf{x}), \quad (17)$$

and so it follows from (14) that  $\mathbf{A}$  can also be written as

$$\mathbf{A} = (\nabla \mathbf{y})^T (\nabla_s \nu)^2 (\nabla \mathbf{y}). \quad (18)$$

Here and in the following, for brevity, we drop the argument from  $\mathbf{A}(\mathbf{y})$  and denote  $\mathbf{A}^* := \mathbf{A}(\mathbf{y}^*)$ .

An interesting interpretation for  $\mathbf{A}$  and its invariance under bending-neutral deformations can be given starting from (18). This can indeed be seen as the referential representation for the *third* fundamental form of  $\mathcal{S}$  (see, for example, p. 205 of ref. 32). Given two arbitrary vectors  $\mathbf{v}_1, \mathbf{v}_2$  on  $S$ ,

$$\mathbf{v}_1 \cdot \mathbf{A} \mathbf{v}_2 = \mathbf{v}'_1 \cdot (\nabla_s \nu)^2 \mathbf{v}'_2 = \mathbf{v}_1 \cdot \mathbf{A}^* \mathbf{v}_2 = \mathbf{v}^*_1 \cdot (\nabla_s \nu^*)^2 \mathbf{v}^*_2, \quad (19)$$

where  $\mathbf{v}'_i = (\nabla \mathbf{y})\mathbf{v}_i$  are vectors tangent to  $\mathcal{S}$ ,  $\mathbf{v}^*_i = (\nabla \mathbf{y}^*)\mathbf{v}_i$  are vectors tangent to  $\mathcal{S}^*$ , and  $\nabla_s \nu^*$  is the curvature tensor of  $\mathcal{S}^*$ . Identity (19) simply says that the third fundamental form is invariant under a bending-neutral deformation.

As pointed out in ref. 33, the tensorial deformation measure for shells considered in the works of Koiter,<sup>34</sup> Sanders,<sup>31</sup> and Buiansky<sup>30</sup> can be given a coordinate-free expression, which in the case of plates reads simply as

$$\mathbf{K} := (\nabla \mathbf{y})^T (\nabla_s \nu) (\nabla \mathbf{y}). \quad (20)$$

This appears to be the referential representation of the second fundamental form of  $\mathcal{S}$ . Compared to  $\mathbf{A}$ ,  $\mathbf{K}$  is not a pure measure of bending.

<sup>\*\*</sup> Here we follow ref. 30 and 31 in the desire to represent the curvature tensor of a sphere of radius  $R$  as  $\nabla_s \nu = \frac{1}{R} \mathbf{P}(\nu)$ .



The linear invariant of  $\mathbf{A}$ ,  $\text{tr}\mathbf{A}$  (which is quadratic in the principal curvatures of  $\mathcal{S}$ ), has already featured in the literature; it emerged as a strain energy density from a discretized model for membranes and plates<sup>35</sup> and, more recently, also from a dimension reduction to plates.<sup>20</sup> In the latter, the reduced strain energy density also involves the scalar invariants of the tensor  $\mathbf{\Pi} := \frac{1}{2}(\mathbf{V}(\nabla_s \mathbf{v}) + (\nabla_s \mathbf{v})\mathbf{V})$ , where  $\mathbf{V}$  is the stretching tensor of  $\mathbf{y}$  in the present configuration, defined as the positive definite root of  $\mathbf{V}^2 = \mathbf{B}$ . It is a simple exercise to show by example that neither  $\mathbf{\Pi}$  nor its invariants are pure measures of bending according to the definition proposed in this paper.

Finally, the shell model advanced in ref. 36 employs strains that reduce to the eigenvalues of  $\mathbf{A}$  in the case of plates, under the further hypothesis that  $\mathbf{V}$  and  $\nabla_s \nu$  commute, a hypothesis implicit in the symmetric class of deformations adopted in ref. 36.<sup>††</sup>

## 2.2 Impure measures of bending

It is interesting to link the invariants of  $\mathbf{A}$  to other invariant, but impure measures of bending, namely, the mean curvature  $H$  and the Gaussian curvature  $K$  of  $\mathcal{S}$ , defined as

$$H := \frac{1}{2}\text{tr}(\nabla_s \mathbf{v}) \quad \text{and} \quad K := \det(\nabla_s \mathbf{v}). \quad (21)$$

We begin by computing  $\text{tr}\mathbf{A}$ . It readily follows from (18) that

$$\text{tr}\mathbf{A} = \text{tr}(\mathbf{B}(\nabla_s \nu)^2) = 2H \text{tr}(\mathbf{B}\nabla_s \nu) - K \text{tr}\mathbf{B}, \quad (22)$$

where the last equality follows from an identity proven in ref. 37 (see their eqn (14)).

To obtain  $\det\mathbf{A}$ , we start from  $\text{tr}\mathbf{A}^2$ , which by (18) also reads as

$$\text{tr}\mathbf{A}^2 = (\nabla_s \nu)^2 \mathbf{B} \cdot \mathbf{B} (\nabla_s \nu)^2, \quad (23)$$

where  $\cdot$  denotes the inner product of tensors.<sup>‡‡</sup> Letting  $\mathbf{B}$  be represented as

$$\mathbf{B} = B_{11}\mathbf{n}_1 \otimes \mathbf{n}_1 + B_{12}(\mathbf{n}_1 \otimes \mathbf{n}_2 + \mathbf{n}_2 \otimes \mathbf{n}_1) + B_{22}\mathbf{n}_2 \otimes \mathbf{n}_2 \quad (24)$$

in the eigenframe of  $\nabla_s \nu$  in (16), we easily obtain from (23) that

$$\text{tr}\mathbf{A}^2 = \kappa_1^4 B_{11}^2 + \kappa_2^4 B_{22}^2 + 2\kappa_1^2 \kappa_2^2 B_{12}^2 = (\text{tr}\mathbf{A})^2 - 2K^2 \det\mathbf{B}, \quad (25)$$

which on applying the Cayley–Hamilton theorem to  $\mathbf{A}$  implies that

$$\det\mathbf{A} = K^2 \det\mathbf{B} = K^2 \det\mathbf{C}. \quad (26)$$

Thus, eqn (22) and (26) show how pure scalar measures of bending may result from combining measures of stretching and impure measures of bending.

In the special case of an isometry, for which  $\mathbf{C} = \mathbf{I} - \mathbf{e}_3 \otimes \mathbf{e}_3$  and  $\mathbf{B} = \mathbf{P}(\nu)$ ,

$$\text{tr}\mathbf{A} = \kappa_1^2 + \kappa_2^2 = 2C \quad \text{and} \quad \det\mathbf{A} = (\kappa_1 \kappa_2)^2 = K^2, \quad (27)$$

where  $C$  is the Casorati total curvature.<sup>38</sup>

<sup>††</sup> This hypothesis, however, fails to make  $\mathbf{\Pi}$  a pure measure of bending for plates.

<sup>‡‡</sup> For generic tensors  $\mathbf{L}$  and  $\mathbf{M}$ ,  $\mathbf{L} \cdot \mathbf{M} := \text{tr}(\mathbf{L}^T \mathbf{M})$ .

## 2.3 Pure rod-bending measure

To connect  $\mathbf{A}$  to Antman's measure of pure bending for rods,<sup>2</sup> we consider a class of deformations of  $S$  that produce a cylindrical surface, as shown in Fig. 2. More precisely, we take

$$\mathbf{y}(x_1, x_2) = y_1(x_1)\mathbf{e}_1 + x_2\mathbf{e}_2 + y_3(x_1)\mathbf{e}_3, \quad (28)$$

where  $y_1$  and  $y_3$  are scalar functions of class  $C^2$ . From (28) we easily arrive at

$$\nabla \mathbf{y} = y_1' \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + y_3' \mathbf{e}_3 \otimes \mathbf{e}_1, \quad (29)$$

where a prime ' denotes differentiation with respect to  $x_1$ . Thus, by applying (10), we see that

$$\nu = \cos \vartheta \mathbf{e}_3 - \sin \vartheta \mathbf{e}_1 \quad (30)$$

$$\text{with} \quad \cos \vartheta = \frac{y_1'}{\sqrt{y_1'^2 + y_3'^2}} \quad \text{and} \quad \sin \vartheta = \frac{y_3'}{\sqrt{y_1'^2 + y_3'^2}}.$$

With this choice of angle  $\vartheta$ , the unit tangent  $\mathbf{t}$  to the curve in space that sweeps  $\mathcal{S}$  along the generatrix  $\mathbf{e}_2$  is represented as

$$\mathbf{t} = \frac{(\nabla \mathbf{y})\mathbf{e}_1}{|(\nabla \mathbf{y})\mathbf{e}_1|} = \cos \vartheta \mathbf{e}_1 + \sin \vartheta \mathbf{e}_3, \quad (31)$$

and it follows from (30) that

$$\mathbf{A} = \vartheta'^2 \mathbf{e}_1 \otimes \mathbf{e}_1. \quad (32)$$

The only non-vanishing invariant of  $\mathbf{A}$  is  $\text{tr}\mathbf{A} = \vartheta'^2$ , which, to within an elastic modulus, is precisely Antman's bending energy for an extensible elastic rod.

## 2.4 An illustration: axisymmetric deformations

To illustrate the theory proposed in this paper, we consider now the general class of axisymmetric deformations of a disk of radius  $R$ . The geometric setting is shown in Fig. 3, where  $S$  is the undeformed disk and  $\mathcal{S}$  is an axisymmetric *nightcap* generated by the deformation  $\mathbf{y}$  parametrized as

$$\mathbf{y}(s, \varphi) = r(s)\mathbf{e}_r + z(s)\mathbf{e}_3, \quad (33)$$

where  $(s, \varphi)$ , with  $0 \leq s \leq R$  and  $0 \leq \varphi \leq 2\pi$ , are polar coordinates in the plane  $(x_1, x_2)$ , so that  $x_1 = s \cos \varphi$  and  $x_2 = s \sin \varphi$ ,  $\mathbf{e}_r$  is the radial unit vector and  $\mathbf{e}_\varphi$  the azimuthal unit vector,  $r(s)$  and  $z(s)$  are smooth functions representing radial

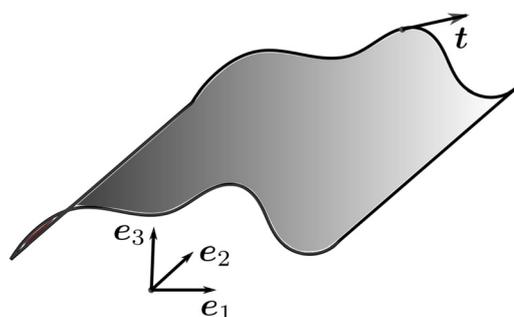


Fig. 2 Cylindrical surface produced by the deformation  $\mathbf{y}$  described in (28).



and vertical components of the image under  $\mathbf{y}$  of the generic point  $\mathbf{x} = s\mathbf{e}_r$  of  $S$ .

Let  $\mathbf{x}(t) = s(t)\mathbf{e}_r(\varphi(t))$  be a generic curve on  $S$  parameterized in  $t$ . Denoting by a superimposed dot differentiation with respect to  $t$ , we readily obtain that

$$\dot{\mathbf{x}} = \dot{s}\mathbf{e}_r + s\dot{\varphi}\mathbf{e}_\varphi \quad \text{and} \quad \dot{\mathbf{y}} = \dot{s}(r'\mathbf{e}_r + z'\mathbf{e}_3) + r\dot{\varphi}\mathbf{e}_\varphi, \quad (34)$$

where a prime  $'$  denotes differentiation with respect to  $s$  and  $\dot{\mathbf{y}}$  is the tangent to the curve  $\mathbf{y}(t)$  resulting from composing  $\mathbf{y}$  with  $\mathbf{x}(t)$ . The deformation gradient  $\nabla\mathbf{y}$  at a point  $\mathbf{x}_0$  must satisfy the identity  $\dot{\mathbf{y}} = (\nabla\mathbf{y})\dot{\mathbf{x}}$  for every curve  $\mathbf{x}(t)$  through  $\mathbf{x}_0$ . The following formula is then an easy consequence of (34),

$$\nabla\mathbf{y} = r'\mathbf{e}_r \otimes \mathbf{e}_r + \frac{r}{s}\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + z'\mathbf{e}_3 \otimes \mathbf{e}_3. \quad (35)$$

It follows from (35) that

$$\mathbf{C} = (\nabla\mathbf{y})^T(\nabla\mathbf{y}) = \lambda_1^2\mathbf{e}_r \otimes \mathbf{e}_r + \lambda_2^2\mathbf{e}_3 \otimes \mathbf{e}_3, \quad (36)$$

where

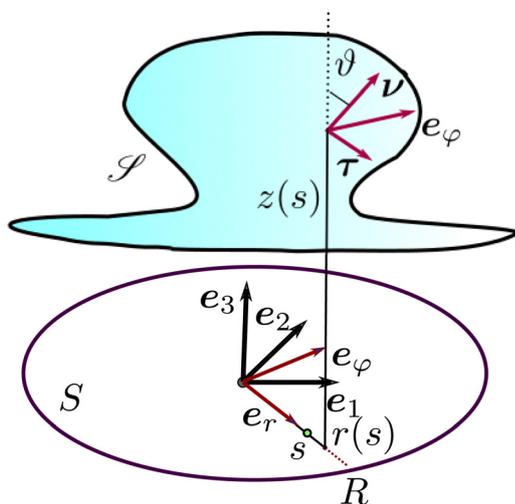
$$\lambda_1 := \sqrt{r'^2 + z'^2} \quad \text{and} \quad \lambda_2 := \frac{r}{s} \quad (37)$$

are the principal stretches of  $\mathbf{U} = \sqrt{\mathbf{C}}$ .

If the top of the nightcap in Fig. 3 is regular, that is, with a well defined tangent plane, symmetry requires the latter to be parallel to the  $(x_1, x_2)$  plane, and so  $\nabla\mathbf{y}$  must vanish there. From (35), this implies that both  $r'$  and  $z'$  must vanish at  $s = 0$  and so does the limiting value of  $r/s$ , meaning that  $\lambda_1(0) = \lambda_2(0) = 0$ . This, in particular, says that the class of axisymmetric deformations  $\mathbf{y}$  of  $S$  cannot accommodate the constraint of inextensibility,  $\lambda_1\lambda_2 \equiv 1$ , unless  $\mathbf{y}$  is singular at the origin.

A simple computation yields

$$\text{co}(\nabla\mathbf{y}) = \frac{r}{s}(r'\mathbf{e}_3 \otimes \mathbf{e}_3 - z'\mathbf{e}_r \otimes \mathbf{e}_3), \quad (38)$$



**Fig. 3** Sketch of an axisymmetric *nightcap*  $\mathcal{S}$  obtained by deforming a flat disk  $S$  of radius  $R$  via the mapping  $\mathbf{y}$  described using (33). The unit vector  $\boldsymbol{\nu}$  is the outward normal to  $\mathcal{S}$ , while  $\boldsymbol{\tau}$  is the tangent unit vector along the meridians of  $\mathcal{S}$ , so that  $(\boldsymbol{\tau}, \mathbf{e}_\varphi, \boldsymbol{\nu})$  is a mobile frame oriented like the fixed frame  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ . The angle  $\vartheta$ , defined in (44), designates the relative inclination of the normals  $\boldsymbol{\nu}$  and  $\mathbf{e}_3$ , to  $\mathcal{S}$  and  $S$ , respectively.

which using (10) leads us to

$$\mathbf{v} = \frac{1}{\lambda_1}(r'\mathbf{e}_3 - z'\mathbf{e}_r). \quad (39)$$

Defining by  $\boldsymbol{\tau} := \mathbf{e}_\varphi \times \boldsymbol{\nu}$  a tangent unit vector to  $\mathcal{S}$  oriented along its meridians so that the mobile frame  $(\boldsymbol{\tau}, \mathbf{e}_\varphi, \boldsymbol{\nu})$  is oriented like  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ , we can write

$$\boldsymbol{\tau} = \frac{1}{\lambda_1}(r'\mathbf{e}_r + z'\mathbf{e}_3), \quad (40)$$

and (35) becomes

$$\nabla\mathbf{y} = \lambda_1\boldsymbol{\tau} \otimes \mathbf{e}_r + \lambda_2\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi. \quad (41)$$

From the latter, we easily compute

$$\mathbf{B} = (\nabla\mathbf{y})(\nabla\mathbf{y})^T = \lambda_1^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} + \lambda_2^2\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi \quad (42)$$

and

$$(\nabla\mathbf{y})^{-1} = \frac{1}{\lambda_1}\mathbf{e}_r \otimes \boldsymbol{\tau} + \frac{1}{\lambda_2}\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi. \quad (43)$$

By introducing the angle

$$\vartheta(s) := -\arctan \frac{z'}{r'}, \quad (44)$$

we give a more compact representation of  $\boldsymbol{\nu}$  and  $\boldsymbol{\tau}$ ,

$$\boldsymbol{\nu} = \cos \vartheta \mathbf{e}_3 + \sin \vartheta \mathbf{e}_r, \quad \boldsymbol{\tau} = \cos \vartheta \mathbf{e}_r - \sin \vartheta \mathbf{e}_3. \quad (45)$$

As a consequence of (45) and (41), we can write

$$\nabla\mathbf{v} = \vartheta' \boldsymbol{\tau} \otimes \mathbf{e}_r + \frac{1}{s} \sin \vartheta \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi, \quad (46)$$

from which it follows that the pure measure of bending  $\mathbf{A}$  in (14) acquires here the explicit representation

$$\mathbf{A} = \vartheta'^2 \mathbf{e}_r \otimes \mathbf{e}_r + \frac{1}{s^2} \sin^2 \vartheta \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi, \quad (47)$$

so that a quadratic pure bending invariant would be

$$\text{tr } \mathbf{A} = \vartheta'^2 + \frac{1}{s^2} \sin^2 \vartheta. \quad (48)$$

For a comparison, we also compute the quadratic invariants of the curvature tensor  $\nabla_s \boldsymbol{\nu}$ . By (17), (43), and (46), we obtain that

$$\nabla_s \boldsymbol{\nu} = (\nabla\mathbf{v})(\nabla\mathbf{y})^{-1} = \frac{1}{\lambda_1} \vartheta' \boldsymbol{\tau} \otimes \boldsymbol{\tau} + \frac{1}{\lambda_2 s} \sin \vartheta \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi, \quad (49)$$

which, compared with (16), yields

$$\kappa_1 = \frac{\vartheta'}{\sqrt{r'^2 + z'^2}} \quad \text{and} \quad \kappa_2 = \frac{\sin \vartheta}{r}. \quad (50)$$

The quadratic impure bending measures would then read as

$$H^2 = \frac{1}{4} \left( \frac{\vartheta'}{\sqrt{r'^2 + z'^2}} + \frac{\sin \vartheta}{r} \right)^2 \quad \text{and} \quad K = \frac{\vartheta' \sin \vartheta}{r \sqrt{r'^2 + z'^2}}. \quad (51)$$

By combining together (42), (47), and the expression for  $K$  in (51), one easily sees that identity (26) is indeed satisfied.

A simple direct theory of isotropic soft plates developed according to the principles purported in this paper would be



based on the following strain energy density (per unit reference area),

$$W := \frac{1}{2}\alpha \operatorname{tr} \mathbf{A} + \frac{1}{2}\beta [\operatorname{tr} \mathbf{U}^2 + 2(1 - \operatorname{tr} \mathbf{U})] \\ = \frac{1}{2}\alpha \left( \vartheta'^2 + \frac{1}{s^2} \sin^2 \vartheta \right) + \frac{1}{2}\beta [(\lambda_1 - 1)^2 + (\lambda_2 - 1)^2], \quad (52)$$

where  $\alpha$  and  $\beta$  are positive elastic moduli. A similar (but not identical) expression for  $W$  amenable to a formal Hamiltonian treatment was proposed for axisymmetric shells<sup>39</sup> (see also p. 642 of ref. 40). The strain energy functional resulting from  $W$  is then

$$F[r, z] := \int_S W \, dA \\ = \pi \int_0^R \left\{ \alpha \left( \vartheta'^2 + \frac{1}{s^2} \sin^2 \vartheta \right) + \beta [(\lambda_1 - 1)^2 + (\lambda_2 - 1)^2] \right\} s \, ds, \quad (53)$$

subject to appropriate boundary conditions on the edge of  $S$ .

By contrast, a quadratic theory building on the impure measures in (51), would be based on an Helfrich-type functional,

$$F_H[r, z] := \int_{\mathcal{S}} (\alpha H^2 + \beta K) \, da \\ = \int_S \lambda_1 \lambda_2 \left[ \frac{1}{4} \alpha \left( \frac{\vartheta'}{\lambda_1} + \frac{\sin \vartheta}{r} \right)^2 + \beta \frac{\vartheta' \sin \vartheta}{\lambda_1 r} \right] \, dA \\ = \pi \int_0^R \left[ \frac{1}{2} \alpha \left( \frac{r}{\sqrt{r'^2 + z'^2}} \vartheta'^2 + \frac{\sqrt{r'^2 + z'^2}}{r} \sin^2 \vartheta \right) \right. \\ \left. + (2\beta + \alpha) \vartheta' \sin \vartheta \right] \, ds, \quad (54)$$

where the area element  $da$  on the present shape  $\mathcal{S}$  has been related to the area element  $dA = s \, ds \, d\varphi$  on the reference shape  $S$  through the equation  $da = \lambda_1 \lambda_2 \, dA$ .

Two features deserve notice in (54). First, as appropriate for a liquid membrane,  $F_H$  is formulated as an integral on the present configuration  $\mathcal{S}$ . Second, the Gaussian curvature  $K$  reveals itself as a *null Lagrangian*, as its contribution to the total strain energy depends only on the boundary value of  $\vartheta$ ,

$$\int_{\mathcal{S}} K \, da = 2\pi [1 - \cos \vartheta(R)], \quad (55)$$

thus becoming an effective edge energy.§§

### 3 Bending-neutrality compatibility

For  $\mathbf{y}'$  to be a bending-neutral deformation, neither  $\mathbf{U}'$  nor  $\mathbf{R}_\nu$  can be arbitrary. They must be subject to a compatibility condition, which we shall discuss in some detail here along with its solutions. The special case where  $\mathbf{U}' = \mathbf{P}(\nu)$  has been considered and fully solved in ref. 23.

§§ The interplay between  $F_H$  and edge strain energies has been explored in a series of papers, some of which are recent.<sup>41–49</sup>

The condition of compatibility arises from the requirement that the curvature tensor  $\nabla_s \nu^*$  of  $\mathcal{S}^*$  be symmetric, as it is expected to be.<sup>23</sup> By writing (17) for  $\nu^*$  (and  $\mathbf{y}^*$ ) instead of  $\nu$  (and  $\mathbf{y}$ ), we have that

$$\nabla \nu^* = (\nabla_s \nu^*)(\nabla \mathbf{y}^*). \quad (56)$$

On the other hand, using (12) and (13), we can also write

$$\nabla \nu^* = \mathbf{R}_0 \nabla \nu = \mathbf{R}_0 (\nabla_s \nu)(\nabla \mathbf{y}), \quad (57)$$

which, combined with (56), leads us to

$$\nabla_s \nu^* = \mathbf{R}_0 (\nabla_s \nu)(\mathbf{U}')^{-1} \mathbf{R}_\nu^T \mathbf{R}_0^T, \quad (58)$$

once use is also made of (6), (3), and (8). It follows from (58) that  $\nabla_s \nu^*$  is symmetric if and only if  $(\nabla_s \nu)(\mathbf{U}')^{-1} \mathbf{R}_\nu^T$  is so. This latter condition will be written in the following equivalent way,

$$\nabla_s \nu = \mathbf{R}_\nu (\mathbf{U}')^{-1} (\nabla_s \nu) \mathbf{R}_\nu \mathbf{U}'. \quad (59)$$

Tensors on both sides of (59) act on the local tangent plane of  $\mathcal{S}$ . It is clear that every solution  $\mathbf{U}'$  of (59) is defined within a multiplicative surface dilation,  $\lambda' \mathbf{P}(\nu)$ , with arbitrary  $\lambda' > 0$ .

Two special classes of solutions of (59) deserve notice. They are better illustrated by using the classical Euler–Rodriguez formula to represent  $\text{SO}(\nu)$  (see also ref. 50 for an updated account),

$$\mathbf{R}(\alpha) := \mathbf{I} + \sin \alpha \mathbf{W}(\nu) - (1 - \cos \alpha) \mathbf{P}(\nu), \quad \alpha \in [-\pi, \pi], \quad (60)$$

where  $\mathbf{W}(\nu)$  is the skew-symmetric tensor associated with  $\nu$ , so that  $\mathbf{W}(\nu)\mathbf{u} = \nu \times \mathbf{u}$ , for all vectors  $\mathbf{u}$ .

First, if either  $\mathbf{R}_\nu = \mathbf{R}(0) = \mathbf{I}$  or  $\mathbf{R}_\nu = \mathbf{R}(\pi) = -\mathbf{I} + 2\nu \otimes \nu$ , the latter acting as  $-\mathbf{I}$  on the plane tangent to  $\mathcal{S}$ , then (59) reduces to the requirement that  $\mathbf{U}'$  commute with  $\nabla_s \nu$ , which just amounts to say that  $\nabla_s \nu$  and  $\mathbf{U}'$  must have the same eigenframe.<sup>22</sup>

Second, for  $\mathbf{U}' = \lambda' \mathbf{P}(\nu)$ , with  $\lambda' > 0$ , (59) becomes

$$\nabla_s \nu = \mathbf{R}_\nu (\nabla_s \nu) \mathbf{R}_\nu. \quad (61)$$

Taking the trace of both sides of (61), and using both (16) and (60), we arrive at

$$2H(1 - \cos 2\alpha) = 0, \quad (62)$$

which for  $2H = \operatorname{tr}(\nabla_s \nu) \neq 0$  reduces to either  $\alpha = 0$  or  $\alpha = \pi$ , delivering again the first special solution of (59) considered above. For  $H = 0$ , which is the case of minimal surfaces, (62) is identically satisfied and direct inspection of (61) with the aid of (60) shows that it is solved by all  $\mathbf{R}_\nu \in \text{SO}(\nu)$ . For  $\lambda' = 1$ , this result was already proved in ref. 23.¶¶

To find the general solution of (59), we represent  $\mathbf{U}'$  in (4) in the eigenframe of  $\nabla_s \nu$  in (16), by writing

$$\mathbf{U}' = \mathbf{R}(\chi) \mathbf{U}'_0 \mathbf{R}(-\chi), \quad (63)$$

¶¶ As appropriately remarked,<sup>23</sup> the same result had also been proved, albeit with different methods, in the differential geometry literature,<sup>51–53</sup> but it was unknown to plates and shells practitioners.



where  $\mathbf{R}(-\chi) = \mathbf{R}(\chi)^T$ ,

$$\mathbf{U}'_0 := \lambda'_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \lambda'_2 \mathbf{n}_2 \otimes \mathbf{n}_2, \quad (64)$$

and  $\chi$  is the angle that  $\mathbf{u}'_1$  makes with  $\mathbf{n}_1$  (the same as the angle that  $\mathbf{u}'_2$  makes with  $\mathbf{n}_2$ ). By using these representation formulae and (60), we finally obtain (59) in the following expanded form,

$$\nabla_s \mathbf{v} = \mathbf{R}(\alpha + \chi) \left( \mathbf{U}'_0 \right)^{-1} \mathbf{R}(-\chi) (\nabla_s \mathbf{v}) \mathbf{R}(\alpha + \chi) \mathbf{U}'_0 \mathbf{R}(-\chi), \quad (65)$$

which, for given  $\kappa_1, \kappa_2$ , is meant to be an equation for  $\lambda'_2/\lambda'_1, \alpha$ , and  $\chi$ .

Actually, it is not difficult to show that (65) amounts to four scalar equations, whose solution is

$$\frac{\lambda'_2}{\lambda'_1} = 1 - \frac{(\kappa_1 + \kappa_2) \sin \alpha}{\kappa_2 \sin \alpha + (\kappa_1 - \kappa_2) \cos \chi \sin(\alpha + \chi)}. \quad (66)$$

Requiring the ratio  $\lambda'_2/\lambda'_1$  as delivered by (66) to be positive, with little labour we arrive at the inequalities

$$\alpha_m < \alpha < \alpha_M, \quad (67)$$

where  $\alpha_m(\chi) := \min\{\alpha_1, \alpha_2\}$  and  $\alpha_M(\chi) := \max\{\alpha_1, \alpha_2\}$ , with

$$\begin{aligned} \alpha_1 &:= \arctan \left( \frac{(\kappa_2 - \kappa_1) \cos \chi \sin \chi}{\kappa_1 \sin^2 \chi + \kappa_2 \cos^2 \chi} \right), \\ \alpha_2 &:= \arctan \left( \frac{(\kappa_2 - \kappa_1) \cos \chi \sin \chi}{\kappa_1 \cos^2 \chi + \kappa_2 \sin^2 \chi} \right). \end{aligned} \quad (68)$$

Thus, for a given  $\chi \in [-\pi, \pi]$  and  $\alpha \in [\alpha_m, \alpha_M]$ ,  $\mathbf{U}'$  is determined using (63) up to a surface dilation  $\lambda' \mathbf{P}(\nu)$ ; in general, there exists a whole three-dimensional set of solutions to (59), parameterized by  $(\chi, \alpha, \lambda')$ . A singular case arises at umbilical points of  $\mathcal{S}$ , where  $\kappa_1 = \kappa_2$ . There, (66) would be incompatible for any  $\alpha \notin \{0, \pi\}$  since it requires  $\lambda'_2/\lambda'_1 = -1$ , whereas, from (68),  $\alpha_m = \alpha_M = 0$ , so that the standard solution  $\mathbf{U}' = \lambda' \mathbf{P}(\nu)$ ,  $\mathbf{R}_\nu = \mathbf{I}$  is recovered by continuity.

A particular two-dimensional family of solutions deserves attention. It is obtained from (66) by setting  $\chi = 0$ , under the assumption that  $\alpha \notin \{0, \pi\}$  (an equivalent family would be obtained by setting  $\chi = \frac{\pi}{2}$ ). If  $\mathcal{S}$  is a hyperbolic surface, that is, if  $K < 0$ , (66) readily gives

$$\frac{\lambda'_2}{\lambda'_1} = -\frac{\kappa_2}{\kappa_1}, \quad (69)$$

while  $\alpha$  is arbitrary. What makes this solution special is the curvature tensor of the surface  $\mathcal{S}^*$  delivered by  $\mathbf{y}'$ . Letting  $\mathbf{R}_0 = \mathbf{I}$ , with no prejudice to generality, we obtain from (58) that

$$\nabla_s \mathbf{v}^* = \frac{\kappa_1}{\lambda'_1} \mathbf{n}_1 \otimes \mathbf{R}(\alpha) \mathbf{n}_1 + \frac{\kappa_2}{\lambda'_2} \mathbf{n}_2 \otimes \mathbf{R}(\alpha) \mathbf{n}_2, \quad (70)$$

whence, also from (69), it follows that

$$2H^* := \text{tr}(\nabla_s \mathbf{v}^*) = \cos \alpha \left( \frac{\kappa_1}{\lambda'_1} + \frac{\kappa_2}{\lambda'_2} \right) = 0, \quad (71a)$$

$$K^* := \det(\nabla_s \mathbf{v}^*) = - \left( \frac{\kappa_1}{\lambda'_1} \right)^2 = - \left( \frac{\kappa_2}{\lambda'_2} \right)^2. \quad (71b)$$

In particular, (71a) implies that  $\mathcal{S}^*$  is a minimal surface.

Clearly, this conclusion relies on the very existence of  $\mathcal{S}^*$ , which would be guaranteed, at least locally, if the surface tensor field defined on  $\mathcal{S}$  as

$$\mathbf{S} := \lambda' \mathbf{R}(\alpha) \mathbf{U}' \quad (72)$$

were a surface gradient. This is an integrability requirement that poses restrictions on both  $\lambda'$  and  $\alpha$ , which remain the only unknown scalar surface fields, once (69) is enforced. The study of these restrictions goes beyond the scope of this paper and will be the subject of a future study, as will also be the integrability of  $\mathbf{S}$  for the general class of incremental deformations that obey (66).

We have considered two distinct conditions that must be met for  $\mathbf{y}'$  to be a bending-neutral deformation, one compatibility condition and one integrability condition. Whenever  $\mathcal{S}$  is such that either one or the other condition is not met, any incremental  $\mathbf{y}'$  will necessarily bring additional bending.

## 4 Conclusions

The primary objective of this paper is to answer a question posed by Wood and Hanna:<sup>1</sup> ‘‘How do we wish to define a measure of bending, and by extension a bending energy and the concept of pure stretching deformation, for a thin structure?’’

We (the patient reader and I) caracolled our way toward a possible answer through the notion of bending-neutral deformation, an incremental deformation that does not affect bending, but it is more than a pure stretching as it may include normal twisting.

Quite naturally, a pure measure of bending was then introduced as a deformation measure unaffected by bending-neutral deformations. An explicit, tensorial example of such a measure was given and contrasted against other measures known for plates.

Common wisdom suggests that plates (and shells, for that matter) are rather unresponsive to twists that induce local drill rotations.<sup>22</sup> This observation may hold true for conventional, *hard* plates, but it may not apply to *soft*, polymeric plates, especially those with an activable internal structure, such as nematic elastomers (described in the landmark textbook by Warner and Terentjev<sup>54</sup>).

These latter material surfaces are *anisotropic*, as the nematic director field  $\mathbf{n}$  on  $\mathcal{S}$  breaks the rotational symmetry about the normal  $\nu$ . It might be interesting to find anisotropic pure measure of bending fit for these systems; they might be suggestive of direct theories where stretching and bending energy contents are kept separate, oblivious to whether they can be derived through dimension reduction of established three-dimensional strain energies or not. |||

Whether the answer presented in this paper to the opening question is satisfactory or not is not for me to tell. In the second

||| The role of bending energy in the theory for nematic elastomer plates is highlighted in the literature<sup>55</sup> and applications have also been presented.<sup>56–58</sup>



part of the paper, my interest shifts toward the legitimacy of bending-neutral deformations. They have been characterized by the fulfillment of a necessary kinematic compatibility condition, which generates a three-dimensional set of solutions. Of course, these only represent *admissible* bending-neutral deformations; which ones do actually exist for a given surface  $\mathcal{S}$ , is decided, at least locally, by the fulfillment of an integrability condition. The study of this condition and its solutions will be the subject of a future paper.

## Conflicts of interest

There are no conflicts to declare.

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