Yield stress fluids and fundamental particle statistics

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Yield stress fluids form a particular state of matter,1 displaying non-linear and novel visco-plasto-elastic flow dynamics upon different boundary conditions. As their name says, they don’t flow until a certain load, the so-called yield stress (or point, \( \tau_0 \)), is applied. This value may be generally interpreted as a shear stress threshold for the breakage of interparticle connectivity.2 Furthermore, as it initiates motion in the system, it is connected to mechanical inertia3 and particle settling, i.e. it is a terse summary of buoyancy, dynamic pressure, weight, viscous and yield stress resistances.4 For prototype systems such as colloids dispersed in a liquid, yield points sensibly depend on the mechanism by which the solid phase tends to interact or aggregate.5–8 The macroscopic constitutive equations they obey, such as the Herschel–Bulkley model, were shown to correspond, over a four-decade range of shear rates, to the local rheological response.9

From the side of an experimenter, however, unambiguously defining a yield stress may not always be straightforward. It can be affected by the experimental procedure adopted, always considering a measurement or some extrapolation technique with the limit of zero shear. Conversely, unyielded domains may be defined by areas where the shear stress second invariant falls below the yield value, plus some small semi-heuristic constant.10 In addition, theoretically, the meaning of notions like \( \tau_0 \) and rheological yielding were questioned to be only qualitative or even to stand for an apparent quantity.11 The dependence they generally show on timescales characteristic of the applied (mechanical) disturbance, also suggested an intimate relationship12 between yield stress and dispersion thixotropy.13 On the other hand, assigning a hydrodynamic or mechanical state below the yield point to a material that is not flowing seems not to be scientifically sound. Experimental values are normally obtained by extrapolation of limited data, whereas careful measurements below the yield point would actually imply that flow takes place.14

At any rate, the analysis of properly defined \( \tau_0 \) concepts forms the subject of interesting investigations and is still a powerful tool in many applications, including macromolecular suspensions,15 gels, colloidal gels and organogels,16–18 foams, emulsions and soft glassy materials.19 It allows for effective comparisons between the resistances which fluids initially oppose to the shear perturbation, somehow specifying a measure of the particle aggregation states taking place in a given dispersant. Electro rheological materials, for instance, exhibit a transition from liquid-like to solid-like behaviors, which is often examined by a yield stress investigation upon a given fluid model (e.g. the Bingham model or the Casson model).20–21 The combination of yield stress measurements with AFM techniques can be used to well-characterize the nature of weak particle attractions and surface forces at nN scales.8 Further issues of a more geometrical nature, which naturally connect to \( \tau_0 \), are rheological percolation22 and its differences from other connectivity phenomena, such as the onset of electric23 or elastic percolation.24,25 In granular fluids, it relates with the theory of jammed states,26 originally pioneered by Edwards.27

In nanoscience as well, the stability control and characterization in single and mixed dispersions or melts is an important and complex step.28,29 Carbon nanotube suspensions,30 for
example, can be prepared in association with other molecular systems, like surfactants and polymers\textsuperscript{31–33} or by (either covalent or non-covalent) functionalization of their walls with reactive groups, which increases the chemical affinity with dispersing agents.\textsuperscript{34} As a consequence of large molecular aspect ratios and significant van der Waals’s attractions, the nanotube aggregation is highly enhanced, giving rise to strongly anisotropic systems of crystalline ropes and entangled network bundles, which are difficult to exfoliate, suspend or even characterize.\textsuperscript{35} Stable CNT dispersions of controlled molecular mass may also exhibit polymeric behavior, and be quantitatively studied by equations taken from the well-established science of macromolecules.\textsuperscript{36,37}

This paper puts forward a basic approach, mostly focused on equilibrium arguments, to devise a yield stress law connected with particle statistics. By conjecturing an ensemble of effective volumes ‘displaced’ at the incipient state of motion, a statistical mechanics picture of $\tau_0$ is proposed. This affords a phenomenological hypothesis that can be developed with reasonable simplicity. The derived relations are applied to typical disperse systems in colloid science and soft matter, such as aqueous and nonaqueous suspensions of ceramic/metal oxides and nanoparticles.

### Defining an ensemble of volumes at the incipient motion

Yield stresses can be generally written as a sum of pairwise bonding contributions, where each particle pair is assigned a larger volume than the juxtaposition of the two initial units,\textsuperscript{38–39} or by averaging the geometric part of the Hamaker expression over a representative pairwise cell.\textsuperscript{40} This analysis relies instead on a statistical definition of an effective volume concept ($V_d$) in a thought experiment. We suppose that, in yielding a dispersion, a canonical ensemble of volumes is displaced from the rest configuration and fulfills well defined statistical laws in thermal equilibrium, at a constant particle number and dispersion volume. The energy perturbation at the onset of motion will diminish with an increasing solid fraction in a representative large cell $V_d$, reflecting an increase in $\tau_0$.

The disperse system will be conjectured to consist of an ensemble of elementary cells, each containing or not containing at least an aggregated solid unit (onward referred to as the “cluster” = cell + aggregates). An equilibrium total cluster number then can be identified with a conserved sum of aleatory variables ($N_k$):

$$n = \sum_k N_k$$  \hspace{1cm} (1)

whose values specify the aggregation statistics in each cell $k$. In the simplest limit cases, one may assign (A) a two-valued set, $N_i = \{0,1\}$, meaning that the $i$-th cell will be respectively empty or occupied by an aggregate, or (B) it may take any integer value, $N_i = \{0,1, \ldots \infty\}$, defining in the extreme case, a cell that would be capable of hosting an open aggregate number. We complete this definition by a second relationship for the portions of fluid displaced in the experiment, conceived for simplicity as a set of discrete terms obeying the following combination law of volumes:

$$\frac{1}{V_d} = \sum_k \rho_k$$  \hspace{1cm} (2)

where $\rho_k = N_k/V_k$ is the $k$-th cluster density, and $V_k$ is the $k$-th contribution to $V_d$. Yield strength will reflect the configuration statistics over the ensemble. Any cell containing aggregates will contribute to it, otherwise it won't affect $\tau_0$ (see Fig. 1). In ‘simple’ yield stress fluids, rest interactions are known to prevent the aggregation structure from breaking as a consequence of thermal agitation, and slow flows display a plastic behavior at very large deformations, without irreversible structural variations.\textsuperscript{1,40} However, the equation systems (1) and (2) will be supposed to generally hold and be adopted irrespective of specific fluid dynamics properties (e.g. thixotropy or shear-thinning, pseudo-plasticity, etc.). Note that the density concept which the first two equations refer to does not coincide with the average dispersion density. The aim is to define an ad hoc (i.e. apparent, quasi-static) solid fraction value by which the yield stress response can be obtained by a statistical mechanics approach at the incipient state of motion.

A criterion assigning a sum over aggregation states is required for a thermodynamic framework, with this being promptly done by the usual partition function concept:

$$Z = \sum_{\{N_k\}} \prod_i P_i$$  \hspace{1cm} (3)

To give a suitable representation of the statistical issue, we define $P_i = P(\nu, V_i)$ to be the probability of finding in $V_i$ an empty (liquid) portion of volume $\nu$ or, equivalently, an empty volume fraction $\psi_i \equiv \nu/V_i$. As we will tackle enough concentrated systems, this choice is suitable for an application of Poisson’s statistics. For spherically symmetric units and negligible excluded volumes, an expression like:

$$P_i \equiv e^{-\psi_i V}$$  \hspace{1cm} (4)

may be adopted,\textsuperscript{41} implying:

$$Z = \sum_{N_1, N_2, \ldots} e^{-\psi_1 V_1 - \psi_2 V_2 - \ldots}$$  \hspace{1cm} (5)

![Fig. 1 Scheme of the fluid cluster structure at the incipient motion. Cell volumes aren't necessarily equal, as depicted here. Those in grey, with cell occupancy larger than zero, contribute to Eqn (2).](image-url)
with normalized canonical probabilities, constrained to eqn (1) and (2):

\[ p(\{\rho_k\}, V_0) = Z^{-1} \prod_i P_i \]  

(6)

The quantity \( \varepsilon \) is introduced as a measure of cluster–cluster interaction strength, and regarded for simplicity as a homogeneous stiffness parameter, independent of \( \rho_k \). The volume fraction \( \psi_i \) is a characteristic of the implied aggregation state.

We will relate the average occupancy in a generic single-particle state to the cluster interaction extent, and thus to \( \tau_0 \). Particle indistinguishability, which is commonly a non-classical feature (but not only, see e.g. ref. 43), will be retained in this framework. From the rules of statistical mechanics, one therefore gets:

\[ \langle N_i \rangle = Z^{-1} \sum_{N_i} \frac{N_i e^{-\varepsilon \psi_i N_i}}{\sum_{N_i} e^{-\varepsilon \psi_i N_i}} \]  

(7)

that, in light of eqn (2), may be rewritten as:

\[ \langle N_i \rangle = \frac{Z_i}{Z} \sum_{N_i} N_i e^{-\varepsilon \psi_i N_i} \]  

(8)

Here, \( Z_i \) denotes the partition function with state \( i \) omitted:

\[ Z_i = \sum_{\{N_{\neq i}\}} \prod_k P_k \]  

(9)

i.e. corresponding to the restricted sum:

\[ n = \sum_{k \neq i} N_k \]  

(10)

Developing eqn (8) returns a distribution of (A) Fermi–Dirac or (B) Bose–Einstein type, and the proof of such a formal analogy is resumed in Appendix 1 upon mapping:

\[ E_k = k_B T \psi_k \]  

(11)

with \( k_B T \) being the Boltzmann thermal energy. This phenomenological equation redefines an effective volume fraction in the displaced fluid (Fig. 2). It has an intuitive significance, as energy perturbations \( (E_k) \) will increase with increasing temperature, stiffness parameter and the velocity perturbation at the onset of motion, reflected by a larger liquid fraction \( \psi_i \), i.e. conditions normally implying a smaller yield stress. In the next section, some observations on fractional statistics, lying between (A) and (B), are reported as well the introduction of an intermediate state (C).

Finally, an equivalence between volume and energy was formerly introduced by Edwards and Oakeshott, although in a different context. Their pioneering theory of powders faced the issue of a statistical mechanics analysis of non-thermal systems, like granular fluids.

### Yield stress and cluster statistics

The most general pressure equation is written as a microscopic average of the stress tensor over the statistical distribution of particle states, provided here by eqn (7). As yield stresses are likewise expected to be linear combinations of \( \langle N_i \rangle \) with unknown (tensor) coefficients, we specialize the calculation to the mean cluster occupancy in the overall representative state conjectured by eqn (2):

\[ \psi = \sum_k \psi_k \]  

(12)

i.e.:

\[ \tau_0 \sim \langle N(\psi) \rangle \]  

(13)

The proportionality constant in this relationship will be removed by forming the experimental quantity \( \tau^* = \tau_0/\tau_{M} \), i.e. dividing by the largest stress within a class of homogeneous and comparable measurements.

In (A), with a two-valued statistics, the arrangement of particle clusters takes the form (Appendix 1):

\[ \langle N_i \rangle = (e^{a \psi_i} + 1)^{-1} \]  

(14)

with \( a \) being a characteristic fluid property, expressible as:

\[ a = -\varepsilon \psi_a \]  

(15)

for some aggregation state denoted by \( \psi_a \). It describes a liquid volume fraction at which the distribution function may either show a phase-like transition or a (rheological) percolation point. Correspondingly, \( \varepsilon \) gives a measure of the average rate at which \( \tau_0 \) is changing near \( \psi_a \). The yield stress value as a function of \( \psi \) thus reads:

\[ \tau_{0,1}(\psi) = \psi(\psi(\psi_a) + 1)^{-1} \]  

(16)

and, as the solid fraction \( \theta \) complementing \( \psi \) obeys:

\[ \psi + \theta = \psi_a + \theta_a \equiv 1 \]  

(17)

the normalized distributions of values in \( \theta \) will scale as:

\[ \tau^*_{\psi}(\theta) = [\tau(\theta) + 1]^{-1} \]  

(18)
still with \( \tau^* = \tau_{0.1}/\tau_{1.1} \), with \( \theta_a \) being a critical threshold/maximum packing density, and:

\[
\xi(\theta) = e^{-\alpha(\theta - \theta_a)}
\]

(19)

In (B), the expression for an infinitely-valued statistic is regained as:

\[
\langle N \rangle = (e^{\alpha+\psi} - 1)^{-1}
\]

(20)

so that:

\[
\tau^*_a(\theta) = [\xi(\theta) - 1]^{-1}
\]

(21)

Note that, as this distribution approaches infinity for \( \xi \rightarrow 1^+ \), it can no longer be normalized to unity. The idealized situation in which cells are permitted to be inde-

regarded.

the remarks in Appendix 2), the solid fraction can be estimated as:

\[
\theta = \langle \Phi^- \rangle
\]

(23)

i.e.:

\[
\langle \Phi^- \rangle = \int_0^1 \Phi^-(k_{1z})P(k_{1z})dy,
\]

(24)

with \( k_{1z}(t) = \sqrt{4\pi}\delta t \) (\( i = d, \) dispersion; \( l, \) liquid) and \( P \) being a Gaussian function. This integral turns out to be well defined, as it is time-independent, and only depends upon \( \phi \) through the ratio \( k_0/k_1 \):

\[
\theta(\phi) = \frac{1}{2} \arctan v
\]

(25)

Here \( v \equiv v_d/m_1 = \eta/\rho \), with shear viscosity and mass density being expressible as \( \eta_d = \eta\eta(\phi) \) and \( \rho_d = \rho\rho(\phi) \), i.e. the product of pure liquid properties times a function of the solid fraction. The reduced quantity \( \bar{v} = v(\phi) \) generally increases with increasing \( \phi \), since viscosity changes should dominate over \( \rho \) values in the concentration regimes of interest. Correspondingly, a larger fluid inertia leads to a reduction of \( \psi \) (eqn (16)), as expected. Note that a critical threshold/maximum packing density \( \theta_a \) cannot be identified from the last equation, being \( \theta \in [0, \frac{1}{2}] \). A boundary/cutoff value \( \theta_a \equiv \phi_a \) needs to be set independently, completing the definition of \( \xi = \xi(\theta) \) in the previous relationships, with the requirement \( \theta < \phi_a \), or:

\[
\bar{v}(\phi) < \tan(\pi\phi_a)
\]

(26)

representing a model constraint for the viscosity, density and the critical volume fraction. Applications of the new equation family will be conducted in conformity with it, as shown in Fig. 7. While eqn (26) always depends on the viscosity model, it is more selective for B and C statistics (\( \alpha < 1 \)), describing steeper liquid–solid transitions.

Eqn (25) redefines an effective volume fraction, \( \phi \rightarrow \theta(\phi) \). To test its validity, experimental measurements from aqueous and non-aqueous systems were taken from the literature. In the first case, the yield stress of ceramic and metal oxide dispersions of Si3N4, α-Ca3(PO4)2, α-ZrO2 and TiO2 (ref. 49) (anatase) were regarded, whereas Al2O3/decalin28 and multiwalled carbon nanotubes/polycarbonate35 (MWCNT/PC melts) were regarded.
as non-aqueous materials. In these systems, interaction mechanisms were mostly London–van der Waals with absent or irrelevant Vold’s effect.\textsuperscript{32} The first three were at/near their isoelectric points, where double-layer surface charges are negligible.\textsuperscript{33} The structure of the anatase colloids was governed too by attractive van der Waals forces, which represents the main interaction mechanism in the last system as well.\textsuperscript{34} Compressive yield stress in the fifth system, alumina in decalin, was still of the van der Waals type, modulated by a steric interparticle repulsive barrier of \(\sim0.7\) nm of propionic acid, with no further electrostatic or structural energies coming into play. With the obvious exception of MWCNT/PC melts \((T = 533\) K), all measurements were conducted at room temperature. The adopted extrapolation laws were Casson or Bingham, unambiguously written in the yield stress, viscosity, and shear rate, with no heuristic constants. In \(\text{Al}_2\text{O}_3\) systems, compressive yield stresses were numerically inferred from the measured variation of volume fraction solids with elevation. Any further physical chemistry details may be found in the references.

Every plot of \(\tau_{1/\alpha}^*\) vs. \(\phi\) was best fitted by means of each particle statistic, \(1/\alpha = 1\) (A), \(\infty\) (B), 2 (C) and the extracted model parameters are depicted in Table 1. Shear viscosity data vs. solid concentration were available for the system MWCNT/PC\textsuperscript{55} and fulfilled Eiler’s law, \(\bar{\eta}(\phi) = \left[1 + \frac{1}{2}[\eta]\phi/(1 - \phi/\phi_a)\right]^2\) (e.g. ref. 56), where a large intrinsic viscosity value \([\eta] = 165\) seems to be a feature of other carbon nanotube suspensions.\textsuperscript{37} Quemada’s model, \(\bar{\eta}(\phi) = (1 - \phi/\phi_a)^{-2}\), was generally adopted in the other systems, still producing a good agreement with yield stress profiles. Examples of the suitability of each model is shown in Fig. 4–6, while Fig. 7 shows how the model constraint in eqn (26) is fulfilled in these cases. Obviously, a similar agreement is also found for \(\text{Ca}_3(\text{PO}_4)_2/\text{H}_2\text{O}\), \(\text{ZrO}_2/\text{H}_2\text{O}\), and \(\text{Al}_2\text{O}_3/\text{C}_{10}\text{H}_{18}\) systems.

Single-valued distribution (A) is the only one predicting a plateau, as is the evident case here of MWCNT/PC melts. A similar situation arises from kaolin colloids in water, paraffinic oil and liquid polybutadiene rubber, for which an S-shaped functional form analogous to eqn (18) was proven to hold.\textsuperscript{58,59} Unluckily, the sparseness of physical chemistry properties of kaolin powders, especially when commercially supplied, do not allow for a data comparison with the other chemical systems, in Tables 1 and 2. Furthermore, the present model turns out to be fundamentally non-linear. Since chemical compositions of kaolin materials are markedly heterogeneous, to average over distinct solid components would require an extension of this approach to the framework of multi-phase media. Finally, on increasing \(\alpha\), (B) and (C) distributions imply steeper behaviors, as they are usually indicative of stronger interactions.

<table>
<thead>
<tr>
<th>Chemical system [s/ l]</th>
<th>(\alpha)</th>
<th>(\phi_a)</th>
<th>(\epsilon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{Si}_3\text{N}_4/\text{H}_2\text{O})</td>
<td>1/2</td>
<td>0.452</td>
<td>31.7</td>
</tr>
<tr>
<td>(\text{Ca}_3(\text{PO}_4)_2/\text{H}_2\text{O})</td>
<td>0</td>
<td>0.509</td>
<td>10.5</td>
</tr>
<tr>
<td>(\text{ZrO}_2/\text{H}_2\text{O})</td>
<td>1/2</td>
<td>0.393</td>
<td>26.1</td>
</tr>
<tr>
<td>(\text{TiO}_2/\text{H}_2\text{O})</td>
<td>0</td>
<td>0.523</td>
<td>18.6</td>
</tr>
<tr>
<td>(\text{Al}<em>2\text{O}<em>3/\text{C}</em>{10}\text{H}</em>{18})</td>
<td>0</td>
<td>0.527</td>
<td>30.9</td>
</tr>
<tr>
<td>(\text{C}/(-(\text{O})-(\text{C})-\text{O})_{n})</td>
<td>1</td>
<td>0.443</td>
<td>69.8</td>
</tr>
</tbody>
</table>

Fig. 4 Example of particle statistics \(\alpha = 0\). Reduced yield stress versus solid volume fraction for the anatase system \(\text{TiO}_2/\text{H}_2\text{O}\) (densely dashed line, \(\alpha = 0\); dashed line, \(\alpha = 1\); solid line, \(\alpha = \frac{1}{2}\)). Model parameters for \(\alpha = 0\) are in Table 1. Best fits with similar quality were also met in \(\text{Ca}_3(\text{PO}_4)_2/\text{H}_2\text{O}\) and \(\text{Al}_2\text{O}_3/\text{C}_{10}\text{H}_{18}\) systems.

Fig. 5 Example of particle statistics \(\alpha = \frac{1}{2}\). Reduced yield stress versus solid volume fraction for \(\text{Si}_3\text{N}_4/\text{H}_2\text{O}\). Lines and symbols are as in Fig. (4), and model parameters for \(\alpha = \frac{1}{2}\) are in Table 1. Best fits with similar quality were also obtained in \(\text{ZrO}_2/\text{H}_2\text{O}\).

Fig. 6 Example of particle statistics \(\alpha = 1\). Reduced yield stress versus solid volume fraction for MWCNT/PC. Lines and symbols are as in Fig. 4, and model parameters for \(\alpha = 1\) are in Table 1.
Improving the agreement in Fig. 6 likely requires the combination of further interaction states.

The maximum solid loading is known exactly in two cases, the first of which, \( \phi_m = 0.54 \) [\( \text{Al}_2\text{O}_3/\text{C}_{16}\text{H}_{16} \)],\(^{39}\) agrees with \( \phi_a = 0.53 \) in Table 1. Volume corrections due to propionic acid layers were estimated to be less than 1%. The second, \( \phi_m = 0.146 \) [\( \text{TiO}_2/\text{H}_2\text{O} \)], was highly affected by porosity (nanoparticles forming an interlinked porous network in the liquid) and thus is not directly comparable with \( \phi_a \). The fraction \( \phi_a \) was reported at the third digit, as the model was sensitive to it.

To interpret the model parameters by the extent of particle interactions, relevant energetic quantities at mesoscopic/macroscopic scales were collected in Table 2, i.e. the Hamaker constant for particles in the dispersant (\( A_H \)), which reflects the difference between the polarizabilities of solid and liquid molecules, and Young’s/bulk moduli of the solid phase \( (E, K) \), defining the elastic response in homogeneous isotropic materials. While \( A_H \) is known to relate with \( \gamma_0 \),\(^{48}\) elastic (and loss) moduli, detected from oscillatory tests at \( \sim 1 \text{ Hz} \), were recently found to be proportional to the yield stress of gel (Carbopol) solutions.\(^{9} \)

The aqueous Hamaker constant was determined by the knowledge of optical spectra of Si\(_3\)N\(_4\) (ref. 60) and ZrO\(_2\) (ref. 61) from Lifshitz theory. For TiO\(_2\), it was deduced from linearly correlating yield stress data with the square zeta potential.\(^{62}\) As in the case of colloidal Al\(_2\)O\(_3\) in decalin,\(^{39}\) the Tabor–Winterton approximation was adopted to evaluate \( A_H \) of tricalcium phosphate particles. Upon neglecting retardation effects, it gives:

\[
A_H = \frac{3}{4} \gamma \left( \frac{\varepsilon_e - \varepsilon_i}{\varepsilon_e + \varepsilon_i} \right)^2 + \frac{3}{16} \frac{\varepsilon_1}{\varepsilon_2} \left( \frac{\varepsilon_e - \varepsilon_i}{\varepsilon_e + \varepsilon_i} \right)^2
\]

with refractive indices in the visible spectrum set to \( n_s = 1.6 \), and \( n_i = 1.33 \) and static dielectric constants \( \varepsilon_e = 15.4 \), and \( \varepsilon_i = 80 \). The UV absorption frequency should range here in (ref. 63) \( (3-4) \times 10^{15} \text{ Hz} \) so that a mean \( \nu_e = 3.5 \times 10^{15} \text{ Hz} \) was assumed. Because data for \( x-\text{Ca}_3(\text{PO}_4)_2 \) are scarce, \( n_s \) is inferred from optical spectra of calcium phosphate glasses\(^{64}\) in the proportion \( [\text{CaO}] : [\text{P}_2\text{O}_5] = 1 : 3 \) and \( n_s \) is that of hydroxyapatite, as it was successfully employed in a DLVO theory for amorphous particles.\(^{65}\) The Hamaker constant for MWCNT/PC was also evaluated, from the combining law\(^{66}\) for the pure materials values \( A_{4H} \approx 100 \text{ zJ} \) (ref. 67) and \( A_{4H} \approx 50.8 \text{ zJ} \),

\[
A_H = \left( A_{4H}^{1/2} - A_{4H}^{1/2} \right)^2
\]

Elastic features were taken from rather recent literature on each of the solid compounds listed in the tables. For Si\(_3\)N\(_4\), an average \( E \) value among three families of samples was considered, which is then representative of Cerademy Ceralloy.\(^{69}\) Near-theoretical density values were adopted for Al\(_2\)O\(_3\) (ref. 69) and \( K \) of \( \beta-\text{Si}_3\text{N}_4 \),\(^{69}\) which was the most abundant silicon nitride phase (–84.4% wt) in the original fluid samples of Fig. 5.\(^{39}\) The Young’s modulus for \( x-\text{Ca}_3(\text{PO}_4)_2 \) was experimentally measured, with the bulk one being calculated instead using \( ab \text{ initio} \) density functional theory calculations (DFT),\(^{71}\) i.e. the same numerical framework employed to get the elastic constants of TiO\(_2\) (anatase)\(^{72}\) and ZrO\(_2\).\(^{73}\) Average values between the predictions from generalized gradient (GGA) and local density approximations (LDA) were regarded for the monoclinic ZrO\(_2\) phase. Concerning MWCNT, \( E \) was taken from an atomistic potential simulation,\(^{74}\) whereas the diamond value was used for \( K \).\(^{75}\)

To proceed, the quantitative insight now is to link \( A_H \) to \( \phi_a \), and \( E \) and \( K \) to \( \varepsilon \). On increasing the extent of the repulsive forces, particles exhibit an effective size \( (r_e) \), and volume fraction \( (\phi) \), relating with unperturbed values as\(^{48}\) \( \phi/\phi^* = (r/r^*)^3 \).

Accordingly, by representing \( \phi_a = \phi_a(A_H) \), one can verify a meaningful linear correlation that extrapolates, in the limit \( A_H \to 0 \), a good estimate for the packing fraction of the simple cubic cell, \( \phi_a (0^+) = 0.517 \) against \( \pi/6 = 0.524 \) (Fig. 8). Regarding \( \varepsilon = \varepsilon(E, K) \), it still shows a reasonable linear trend with both elastic constants (Fig. 9 and 10). As expected upon \( E > 0 \), the stiffness parameter \( \varepsilon (0^+) \to 0 \) as well. The behaviors in Fig. 8–10 did not vary appreciably upon changing the yield stress data from Bingham to Casson, which were both available for anatase.\(^{49}\)

Such correlations, especially with the Young’s modulus, are suggestive of a linear law between \( \varepsilon \) and the interatomic spring constant of the solid compounds, consistent with a microscopic definition of the stiffness parameter, i.e.:

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### Table 2: Particle–particle interactions and elastic solid constants

<table>
<thead>
<tr>
<th>Chemical system (s/l)</th>
<th>( A_H ) [zJ]</th>
<th>( E ) [GPa]</th>
<th>( K ) [GPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Si}_3\text{N}_4/\text{H}_2\text{O} )</td>
<td>46</td>
<td>320</td>
<td>245</td>
</tr>
<tr>
<td>( \text{Ca}_3(\text{PO}_4)_2/\text{H}_2\text{O} )</td>
<td>23</td>
<td>104</td>
<td>76</td>
</tr>
<tr>
<td>( \text{ZrO}_2/\text{H}_2\text{O} )</td>
<td>88</td>
<td>241</td>
<td>181</td>
</tr>
<tr>
<td>( \text{TiO}_2/\text{H}_2\text{O} )</td>
<td>37</td>
<td>167</td>
<td>192</td>
</tr>
<tr>
<td>( \text{Al}<em>2\text{O}<em>3/\text{C}</em>{16}\text{H}</em>{16} )</td>
<td>20</td>
<td>400</td>
<td>255</td>
</tr>
<tr>
<td>( C/(O-(C=O)-O)_n )</td>
<td>8</td>
<td>740</td>
<td>442</td>
</tr>
</tbody>
</table>

This is a table that lists the Hamaker constants \( A_H \) and elastic moduli \( E \) and \( K \) for different chemical systems, as evaluated in the text. The entries are given in zettajoules [zJ], gigapascals [GPa], and terapascals [TPa], respectively. The table highlights the correlation between the Hamaker constant and the elastic moduli of various solid materials, providing insights into their interaction and mechanical properties in colloidal systems.
interaction rate. This doesn’t mean, evidently, that (B) or (C) distributions would not yield good numerical approximations for such cases as well. Second, the model inadequacy to reproduce \( \tau_{1/a} \to 0 \) in the limit of infinite dilution is anyway recovered upon \( \varepsilon \to \infty \). The actual values of stiffness parameters should guarantee a rather fast convergence since, for (A) and (B) distributions with \( \varepsilon^{(\phi_a - \frac{1}{2})} \gg 1 \) and \( \phi_a \sim \frac{1}{2} \),

\[
\tau_{1,s}^* = e^{-\varepsilon/2} \left[ 1 + \frac{\varepsilon}{2\pi (\phi_a)^3} \phi \right] + O(\varepsilon^3),
\]

the numbers in Table 1 reasonably return \( \tau_{1,s}^* \sim (10^{-4} - 10^{-1}) \) when \( \phi \to 0^+ \).

Conclusive remarks

Solid concentration affects the properties of complex fluids, and we put forward the (re)definition of an effective volume fraction (\( \bar{\theta} \)) for the yield stress behavior, here evaluated in terms of a reduced kinematic viscosity. Aggregation clusters contributing to \( \tau_0 \) are modelled by canonical ensembles of (displaced) volumes, with particle statistics determined by the occupancy number.

This conjecture points out an average cluster that is representative of the incipient state of motion, with given liquid and solid fractions. A class of statistical mechanics laws defines yield stress in terms of two coefficients, the maximum packing threshold (\( \phi_a \)) and the particle stiffness parameter (\( \varepsilon \)), e.g.:

\[
1/\tau_0(\phi) \sim e^{-l/\xi(\phi) - \phi} + 1
\]

which turn out to relate to the Hamaker constant and elastic moduli of the solid phase.

Conflicts of interest

There are no conflicts to declare.

Appendixes

Appendix 1

We resume here some of the well known results for eqn (8) in (A) and (B).6 In comparison to the original theories, the energy of the level \( kE_i \) is mapped initially into \( \psi_s \), the Boltzmann energy \( (k_B T) \) into \( 1/\varepsilon \) and the chemical potential (\( \mu \)) into \( \phi_s \). In fact, the energy representation is regained upon eqn (11).

Consider thus the restricted partition function specified by eqn (9) and (10). In (A), the mean particle number in a state \( i \) reads:

\[
<N_i> = \frac{P_i Z_i(n-1)}{Z_i(n) + P_i Z_i(n-1)}
\]

(A.1)

where, to work it out, the following expansion can be used upon \( \Delta n/n \ll 1 \):

\[
\ln Z_i(n - \Delta n) \approx \ln Z_i(n) - a_i \Delta n
\]

(A.2)
and, for a sum over many states, the following approximation:

$$a_i = \left( \frac{\partial \ln Z}{\partial n} \right) = \left( \frac{\partial \ln Z}{\partial n} \right) = a$$ (A.3)

implies:

$$\frac{Z_i(n-1)}{Z_i(n)} = e^{-a}$$ (A.4)

Replacing this result into eqn (A.1) proves eqn (14). In (B), let’s use the former logarithmic expansion:

$$\langle N_i \rangle = \frac{N_i P_i e^{-a N_i} \sum N_j P_j e^{-a N_j}}{\sum N_j P_j e^{-a N_j}} (A.5)$$

to get:

$$\langle N_i \rangle = \frac{N_i P_i e^{-a N_i}}{\sum N_j P_j e^{-a N_j}} (A.6)$$

the numerator of which is expressible through:

$$\sum \sum N_i P_i e^{-a N_i} = -\frac{\partial}{\partial \psi_i} \sum N_i P_i e^{-a N_i}$$ (A.7)

Obviously, after noting the infinite geometric series:

$$\sum N_i P_i e^{-a N_i} = (1 - e^{-\psi_i - a})^{-1}$$ (A.8)

Eqn (20) is recovered at once, since:

$$\langle N_i \rangle = \frac{1}{\varepsilon} \frac{\partial}{\partial \psi_i} \ln (1 - e^{-\psi_i - a})$$ (A.9)

Remember that eqn (A.3) upon eqn (11) gives $a = -\mu(k_B T)$. We accordingly interpret the ratio $-a/\varepsilon$ in the corresponding relationship to assign a characteristic aggregate state of the disperse system (a percolation-like point $\psi_0$) in every statistic here regarded.

**Appendix 2**

For the incompressible flow in Fig. 3, with $\delta u_x = \delta u_z = 0$, with no pressure gradient and volume forces, the balanced equation for the momentum along $x$ brings us to:

$$\rho \frac{\partial \delta u_x}{\partial t} = -\eta \frac{\partial^2 \delta u_x}{\partial r^2}$$ (B.1)

where the constants, $\rho$ and $\eta$, denote the mass density and shear viscosity coefficient. This is the basic equation to get the complementary error function and the Gaussian probability distribution in eqn (24).

To evaluate the fraction $\theta$ in eqn (17), we form the quantity:

$$\theta(r, \delta t) = \frac{\delta X \delta Y \delta Z}{\delta X \delta Y \delta Z}$$ (B.2)

with each displacement at the numerator being time-dependent, with $\delta X = \delta X(0)$. Since no perturbation develops along $y$ and $z$, one has $\delta y/\delta Y = \delta z/\delta Z = 1$, and:

$$\theta(y, \delta t) = \frac{\delta u_x}{\delta U} = \Phi\left( \frac{y}{\sqrt{4\pi\delta t}} \right)$$ (B.3)

where $\delta x/\delta u_x = \delta x/\delta U = \delta t$. To average the solid fraction and eliminate the dependencies on time and space, we limit ourselves to the free momentum distribution ($P$) in the liquid flow and disregard perturbations from long-ranged hydrodynamic interactions among particles. They are expected to be negligible upon decreasing speed at the incipient motion and increasing dilution of the dispersed units.

Hydrodynamic back-flows may be effectively screened as well by charged particles, while in systems like complex fluid interfaces they are generally not.

Therefore, still with the same notations of eqn (24), it turns out that $\langle x_i \rangle = x_i y_z$:

$$\theta = \langle \theta(y, \delta t) \rangle = \int \Phi\left( (k_{\delta y} y) \right) P(x_i \delta x) \, dx$$ (B.4)

which, since:

$$\int_0^\infty P(x_i \delta x) \, dx = 1$$ (B.5)

and:

$$\int_0^\infty \Phi\left( (k_{\delta y} y) \right) P(x_i \delta x) \, dy = -\frac{i}{2\pi} \ln \left( \frac{y + iv_0}{y - iv_0} \right)$$ (B.6)

reduces, after some mathematical developments, to eqn (25).

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**References**

26 P. Ball, Nat. Mater., 2016, 15, 1227.
43 A. Bach, Indistinguishable Classical Particles (Lecture Notes in Physics, Monograph 44), Springer-Verlag, Berlin Heidelberg, New York, Germany, 1997.